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ABSTRACT

The general purposes of a model are discussed, then seven papers are presented which demonstrate several different facets of the problems associated with constructing and using models of mathematics learning. In the first paper, the task addressed concerns how to incorporate developmental psychology into perceptions of what is happening in the mind of the child; in the next paper, children's approaches to problem solving are examined when the given information includes more than is necessary to solve the problem. In two other papers, the mathematics that provides the goals for instruction is considered. In the fifth paper, a model for learning mathematics that is similar to the traditional models used in the physical sciences is presented; the use of clinical interviews in building an adequate model of learning mathematics is discussed in the next paper; and in the final paper, several major principles in a Piagetian-oriented model are examined. The mathematical areas covered by the models include problem solving, geometry, arithmetic computation, counting, and numeration. (DT)

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MODELS
FOR
LEARNING MATHEMATICS

Papers from a Research Workshop

Sponsored by The Georgia Center
for the Study of Learning and
Teaching Mathematics
and the
Department of Mathematics Education
University of Georgia
Athens, Georgia

Alan R. Osborne, Editor
David A. Bradbard, Technical Editor

ERIC Center for Science, Mathematics,
and Environmental Education
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The Ohio State University
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MATHEMATICS EDUCATION REPORTS

The Mathematics Education Reports series makes available recent analyses and syntheses of research and development efforts in mathematics education. We are pleased to make available as part of this series the papers from the Workshop on Number and Measurement Concepts sponsored by the Georgia Center for the Study of Learning and Teaching Mathematics.

Other Mathematics Education Reports make available information concerning mathematics education documents analyzed at the ERIC Information Analysis Center for Science, Mathematics, and Environmental Education. These reports fall into three broad categories. Research reviews summarize and analyze recent research in specific areas of mathematics education. Resource guides identify and analyze materials and references for use by mathematics teachers at all levels. Special bibliographies announce the availability of documents and review the literature in selected interest areas of mathematics education. Reports in each of these categories may also be targeted for specific sub-populations of the mathematics education community.

Priorities for the development of future Mathematics Education Reports are established by the advisory board of the Center, in cooperation with the National Council of Teachers of Mathematics, the Special Interest Group for Research in Mathematics Education, and other professional groups in mathematics education. Individual comments on past Reports and suggestions for future Reports are always welcomed by the ERIC/SMEAC Center.

Jon L. Higgins
Associate Director

Contents

Acknowledgements and Overview
Leslie P. Steffe, Thomas J. Cooney, and Larry L. Hatfield vii

The Use of Models in Mathematics Education
Alan R. Osborne 1

A Constructivist Model for Instruction
Charles D. Smock 7

Problem Solving in a Model for Early Mathematics Learning
Doyal Nelson 31

Linguistic, Logical, and Cognitive Models for Learning Mathematical Concepts
Harry Beilin 71

The Erlanger Programm As a Model of the Child's Construction of Space
J. Larry Martin 91

A Dynamical Model for Cognitive Development with Applications to Problem Solving
Donald G. Saari 121

Learning Difficulties in Children's Arithmetic: A Clinical Cognitive Approach
Herbert Ginsburg 145

On a Model for Teaching Young Children Mathematics
Leslie P. Steffe 173

Acknowledgements and Overview

The Georgia Center for the Study of Learning and Teaching Mathematics (GCSLTM) was started July 1, 1975, through a founding grant from the National Science Foundation. Various activities preceded the founding of the GCSLTM. The most significant was a conference held at Columbia University in October of 1970 on Piagetian Cognitive-Development and Mathematical Education. This conference was directed by the late Myron F. Rosskopf and jointly sponsored by the National Council of Teachers of Mathematics and the Department of Mathematical Education, Teachers College, Columbia University with a grant from the National Science Foundation. Following the October 1970 Conference, Professor Rosskopf spent the winter and spring quarters of 1971 as a visiting professor of Mathematics Education at the University of Georgia. During these two quarters, the editorial work was accomplished on the proceedings of the October conference and a Letter of Intent was filed in February of 1971 with the National Science Foundation to create a Center for Mathematical Education Research and Innovation. Professor Rosskopf's illness and untimely death made it impossible for him to develop the ideas contained in that Letter.

After much discussion among faculty in the Department of Mathematics Education at the University of Georgia, it was clear that a center devoted to the study of mathematics education ought to attack a broader range of problems than was stated in the Letter of Intent. As a result of these discussions, three areas of study were identified as being of primary interest in the initial year of the Georgia Center for the Study of Learning and Teaching Mathematics--Teaching Strategies, Concept Development, and Problem Solving. Thomas J. Cooney assumed directorship of the Teaching Strategies Project, Leslie P. Steffe the Concept Development Project, and Larry L. Hatfield the Problem Solving Project.

The GCSLTM is intended to be a long-term operation with the broad goal of improving mathematics education in elementary and secondary schools. To be effective, it was felt that the Center would have to include mathematics educators with interests commensurate with those of the project areas. Alternative organizational patterns were available--resident scholars, institutional consortia, or individual consortia. The latter organizational pattern was chosen because it was felt maximum participation would be then possible. In order to operationalize a concept of a consortia of individuals, five research workshops were held during the spring of 1975 at the University of Georgia. These workshops were (ordered by dates held) Teaching Strategies, Number and Measurement Concepts, Space and Geometry Concepts, Models for Learning Mathematics, and Problem Solving. Papers were commissioned for each workshop. It was necessary to commission papers for two reasons. First, current analyses and syntheses of the knowledge in the particular areas chosen for investigation were needed. Second, catalysts for further research and development activities were needed--major problems had to be identified in the project areas on which work was needed.

Twelve working groups have emerged from these workshops, three in Teaching Strategies, five in Concept Development, and four in Problem Solving. The three working groups in Teaching Strategies are: Differential Effects of Varying Teaching Strategies, John Dossey, Coordinator; Development of Protocol Materials to Depict Moves and Strategies, Kenneth Retzer, Coordinator; and Investigation of Certain Teacher Behavior That May Be Associated with Effective Teaching, Thomas J. Cooney, Coordinator. The five working groups in Concept Development are: Measurement Concepts, Thomas Romberg, Coordinator; Rational Number Concepts, Thomas Kieren, Coordinator; Cardinal and Ordinal Number Concepts, Leslie P. Steffe, Coordinator; Space and Geometry Concepts, Richard Lesh, Coordinator; and Models for Learning Mathematics, William Geeslin, Coordinator. The four working groups in Problem Solving are: Instruction in the Use of Key Organizers (Single Heuristics), Frank Lester, Coordinator; Instruction Organized to use Heuristics in Combinations, Phillip Smith, Coordinator; Instruction in Problem Solving Strategies, Douglas Grouws, Coordinator; and Task Variables for Problem Solving Research, Gerald Kulm, Coordinator. The twelve working groups are working as units somewhat independently of one another. As research and development emerges from working groups, it is envisioned that some working groups will merge naturally.

The publication program of the Center is of central importance to Center activities. Research and development monographs and school monographs will be issued, when appropriate, by each working group. The school monographs will be written in nontechnical language and are to be aimed at teacher educators and school personnel. Reports of single studies may be also published as technical reports.

All of the above plans and aspirations would not be possible if it were not for the existence of professional mathematics educators with the expertise in and commitment to research and development in mathematics education. The professional commitment of mathematics educators to the betterment of mathematics education in the schools has been vastly underestimated. In fact, the basic premise on which the GCSLTM is predicated is that there are a significant number of professional mathematics educators with a great deal of individual commitment to creative scholarship. There is no attempt on the part of the Center to buy this scholarship--only to stimulate it and provide a setting in which it can flourish.

The Center administration wishes to thank the individuals who wrote the excellent papers for the workshops, the participants who made the workshops possible, and the National Science Foundation for supporting financially the first year of Center operation. Various individuals have provided valuable assistance in preparing the papers given at the workshops for publication. Mr. David Bradbard provided technical editorship; Mrs. Julie Wetherbee, Mrs. Elizabeth Platt, Mrs. Kay Abney, and Mrs. Cheryl Hirstein, proved to be able typists; and Mr. Robert Petty drafted the figures. Mrs. Julie Wetherbee also provided expertise in the daily operation of the Center during its first year. One can only feel grateful for the existence of such capable and hardworking people.

Thomas J. Cooney Director Teaching Strategies	Leslie P. Steffe Director Concept Development and Director, GCSLTM	Larry L. Hatfield Director Problem Solving

The Use of Models in Mathematics Education

Alan R. Osborne, Editor

The Ohio State University

The word "model" signifies an important concept in mathematics education. It is impractical, even impossible, to achieve the romantic ideal of a complete and total description of all of the factors affecting the teaching and learning of mathematics. Researchers and practitioners alike must settle for describing a corner of the reality that is the teaching and learning of mathematics. Since models are used out of necessity, it behooves us to consider carefully the nature of a model, how it is constructed, the pitfalls and payoffs associated with the use of models, as well as what a model orientation can do for the field of mathematics education.

A model serves a variety of purposes in mathematics education. First, a model is a predictive device. Describing a portion of what happens when learning or teaching takes place, it should predict outcomes of that learning or teaching. Second, a model is a thought-stimulating mechanism serving to suggest critical components of the context of learning and teaching. Third, a model facilitates communication among researchers and, more importantly, between researchers and practitioners.

The predictive purpose of a model is at the essence of the nature of a model. This purpose reflects the influences of the more mature sciences on the field of education. The intent is to identify those essential variables, parameters, and conditions in the environment that produce comfortable and efficient learning in mathematics. For some researchers, identification of salient variables provides an end in itself and is sufficient reward for the creative efforts spawned by their curiosity. But the control that this predictive capability brings with the application of the model justifies our attempts at research in mathematics education. If the application of a model yields easier, more efficient, and happier learning by children and more comfortable, rewarding teaching in the schools, then research efforts are perceived as worthwhile by educational policy makers. Thus, within the predictive capability of a model, we find that more is at stake than the goal of one researcher talking to another researcher. At the heart of an orientation to models is the desire to specify functional control over learning and teaching.

The second purpose of a model orientation is in terms of stimulating questions concerning learning and teaching processes. The traditional purpose of research via the scientific method has been described in terms of refining and extending our conception of reality. By asking more precise questions, by being sure not to overlook critical factors in the learning process, and generally attending to the comprehensiveness of the model, the predictive capability of the model can be extended. But a model does more than stimulate questions within the model; it also suggests reconstructing our conception of reality. Implicit within the selection of one model of learning or teaching is the question of whether that model is the most appropriate or the best prediction of reality. The shifting of a paradigm or model has frequently had a salutory effect in both the physical and life sciences. Such shifts or reconstructions of reality have often stimulated new and insightful questions revealing different operant processes that improve the predictive capability within the particular field of science.

The third purpose of using a model is to improve the communication within the field of science. A carefully described model is interpretable to both the researcher and to the practitioner. An adequate description of a model yields a rationale for an experiment. This rationale yields the specification of the variables, the definition of methodology, an indication of the appropriate measures to be taken and a delimitation of many of the significant constraints. In short, it establishes the base necessary for communicating the essentials of the experiment. This base for communication is missing within a laissez-faire open description of research. Thus a researcher can perceive more readily what the experiment or evaluation was about and can adduce more precisely how the findings fit with his own. The ability to conduct meaningful worthwhile replications is increased. The odds for conducting coordinated complementing research studies are improved. One of the real needs in mathematics education, as well as education more generally, is to design experimental programs that add up to something. Unfortunately, most bits of research appear as separate and discrete entities unto themselves. A model for learning and teaching should provide a matrix into which experiments would fit. Research should build to a more complete and comprehensive picture of what happens when an individual learner copes with mathematics. The coordination of research cannot happen without adequate communication of experimental results nor without fitting the experiments into a larger plan for research. A model provides a major vehicle for accomplishing this.

Most of the applications of research results or findings are not by scientists. Rather practitioners and agencies make use of the work of scientists. Most practitioners in education want principles, rules, or maxims for application. An adequate model does provide the setting from which maxims or principles are generated. The model serves as a communication device encapsulating the control features within the model.

We have made several errors in the past in conceiving of models in mathematics education. We have had a rather vacuous fixation on the idea of model. Educators have been prone to connect boxes and circles with lines and arrows, scatter some labels over the network, and call this apparition a model. Having just enough familiarity with the benefits of using models in science, many educators have said, in effect, "Now we are being scientific; isn't that wonderful." Ignoring the hard work that goes into building a model, the careful thought needed to identify the critical variables and to describe and control the environment of experimentation, and the sheer artistry of building a comprehensive model, such people are simply indulging in self aggrandizement.

To be included within such vacuous fixations is the building of models based upon statements so weak that their truth cannot be questioned but which have little predictive power. Thus, to say that learning is a product of the learner, the curriculum, the teacher, and society is to say little in the sense of a model. Of course this is a valid observation. This does not provide the researcher or the practitioner with an appreciable amount of control over learning or very much insight into that learning.

Another category of error made in the name of models in mathematics education is the confusion of research and statistical designs with the model. Thus, some students of learning in mathematics are more concerned with niceties of blocking, cell size, whether to use nonparametric statistics or not, and other questions of this order than what is being learned or taught about mathematics. Such designs and research models are not directed to the learning of mathematics; rather they are tools of research in mathematics learning in the same sense that a thermometer is a tool for a chemist or a differential equation is a tool for a physicist. The design of the research problem should not be confused with the idea of a model in the learning or teaching of mathematics. Mathematics educators must be familiar with these tools if they are to be researchers or consumers of research.

Some professionals in mathematics education have rejected the statistical approach to research and adopted a more clinical approach. For some, this is a mature recognition of the complexity of human learning. The researcher recognizes that there are several levels of reality in the study of the learning of mathematics, each having the potential for revealing aspects about a model of mathematics learning. Most significant problems in mathematics education require study on several levels and with many different tools before one can say with certainty that a particular model provides adequate explanation and predictive power. But for other researchers, the rejection of the statistical approach has led to usage of the clinical approach to justify haphazard research methodology and the statement of isolated unrelated observations as "truth" or "reality." Researchers operating within this latter framework who do not respect and recognize the power (and limitations) of statistical tools are anti-intellectuals. They reject an entire category of needed and useful studies of mathematics learning.

The papers included in this volume exemplify many different means of using models to stimulate questions, to reconstruct our perception of the reality of how learners deal with mathematics, what should go into an adequate research model, and how teachers might use a model. The problems of building and using a model are attacked on several different fronts.

The paper by Charles Smock addresses the task of how to incorporate developmental psychology into our perceptions of what is happening in the mind of the child. Smock makes the critical point that we should not confuse the child's construction of reality with our perceptions and conceptions of what the child is manipulating and thinking. He issues a warning that all who are building models of human thought processes must be aware of this confusion.

The problems that Doyal Nelson reports are fascinating examples of some of the principles espoused by Smock. In each problem, the child is faced with more information than is necessary to solve the problem. These distracting elements represent noise that the child must structure in some way to get to the meat of the problem. In examining this aspect of problem solving, Nelson's research group reports the differential effects of this chaotic world of extra information in terms of the maturity of learners.

Harry Beilin and Larry Martin each deal with the mathematics that provides the goals for instruction. Each approaches the building of a model for learning from the vantage point of the nature of what is to be learned. Each suggests the nature of mathematics must be a significant factor in the design of any powerful model. Beilin discusses different psychologies of learning in building a model in terms of the nature of mathematics. Martin analyzes the use of mathematics to model children's thought. Each indicates that a researcher's philosophy of mathematics is evident in the conception of a model for learning mathematics.

Donald Saari constructs a model for learning mathematics that looks similar to what is now the traditional means of constructing a model in the physical sciences. First, some assumptions concerning the learning process or problem solving are stated. Then these are restated in terms of functions that approximate the assumptions. Thus, a structured space is constructed that is hypothesized to represent learning. The task of the model builder represented here is fitting a perception of learning and learning processess to a mathematical description. The important next step is verifying the "fit." Are the functional equations predictive of learner behavior?

Herbert Ginsburg's paper is a caution to model builders. Frequently researchers base their conclusions on how children perform on tests. Many of these are paper and pencil tests that are at best symptomatic of the thinking of the child in dealing with the mathematics. Thus, Ginsburg's paper indicates a need to examine carefully what children's performance means. His paper suggests that cognitive clinical interviews are one of the most significant means of coping with this problem in building an adequate model of learning of mathematics.

The final paper copes with a significant problem for the researcher who indulges in constructing models. This problem is interpreting a complicated model for the practitioner. We know enough about the learning of mathematics to state that any nontrivial model is complex. Les Steffe attempts to capture several major principles in a Piagetian-oriented model for the practitioner.

These seven papers demonstrate several different facets of the problems associated with constructing and using models of mathematics learning. None attempt to describe the instructional or teacher component of learning to any marked degree, nor does any paper attempt to tightly specify sociological and/or environmental influences on learning. But each does capture a corner of the reality that is learning mathematics.

If the field of mathematics education is to acquire the maturity of a science and an art, it must attend to constructing models of learning and testing them in a reasonable scientific fashion. Replication and fitting pieces of research together is a necessary condition to attain this maturity. Replication and coordination of research is made possible through careful and judicious use of models.

A Constructivist Model for Instruction[1, 2]

Charles D. Smock

University of Georgia

> Science is not just a collection of laws, a catalogue of unrelated facts. It is a creation of the human mind, with its freely invented ideas and concepts . . . . The only justification for our mental structures is whether and in what way our theories form . . . a link with the world of sense impression.
>
> (Einstein)

The fruitfulness of mathematics education research on problems and issues posed by Piaget depends on the collaboration of individuals in several disciplines. Certainly, many problems of concern to mathematics educators are common to those of developmental psychologists--but many are not. At the same time, the identification of psychological issues critical to the mathematics education researcher will determine the degree of progress to be expected from the current work in mathematics education. These meetings provide a forum that will help isolate the common problems of interest in the two disciplines.

The influence of Piaget's work is so pervasive that one hardly knows where to begin. But, why not at the beginning? That is, Piaget's primary concern has been epistemology in the study of the nature of knowledge and knowledge acquisition. If we wish to understand Piaget's theory of cognitive development, and to conduct empirical investigations relevant to the implications of that theory, consideration of his epistemological foundations

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[1] This paper is a specially prepared version of a paper entitled Constructivism and Principles for Instruction (Smock, 1974).

[2] This report is based on activities supported by the Mathemagenic Activities Program-Follow Through, under grant No. OEG-0-8-522478-4617 (287) Department of HEW, U. S. Office of Education. The opinions expressed herein however, do not necessarily reflect the position or policy of the U. S. Office of Education, and no official endorsement by the U. S. Office of Education should be inferred.

is essential. Essential, first because that foundation influences his theoretical constructions relevant to development of concepts of space, time, causality, et cetera. Essential, second because the setting of observational conditions to test that theory must meet certain methodological imperatives if the findings are to provide relevant information and not merely "noise."

We need also to begin with Piaget's own writing and research findings. In doing so, we must go beyond a surface critical analysis (e.g., beyond "obvious" weakness in design or number of subjects) to the structuring and organization of the research as examples of his epistemology and theory in action. I am not proposing that we not be critical--rather that we need to be sure that we do not engage in criticisms that represent the assimilative tendencies of the critic rather than contradictions inherent in the theory itself.

Everybody "knows" what is wrong with education; in fact, the definitions of the problems by professionals are the same definitions of "everyman." In most cases, the solutions proffered by professionals are not only the traditional ones but often more trivial than those of the "amateur." Few have the courage to face two fundamental questions involved: (a) what is the purpose of the institution of public education, and (b) what is the nature of knowledge and knowledge acquisition? Failure to deal with the first leads to tinkering with trivial elements of a complex societal institution that should be a mechanism for guiding change (whether of the child or society) while unwillingness to confront the second produces, at best, temporary excitement about "innovations." There is hope, however; a few scholars are beginning to view the school as an important element in cultural change (Sarason, 1971; Smock, Graham, Silverman, & Huberty, 1975). This paper, I hope, reflects an appropriate interpretation of the implications of one's view of the nature of knowledge acquisition for instruction of young children in mathematics.

Currently, the names Piaget and/or laboratory, especially mathematics laboratory, are sure to attract interest. While "laboratories" are gaining in fad appeal, meaning for such an instructional technique has remained loosely and ambiguously defined. Teachers who attempt to use such a teaching strategy typically have been forced to rely on a set of "rule of thumb" slogans such as: "concrete understanding before abstract;" "intuitive understanding before formalization;" "activities, then pictures, then symbols;" "discovery rather than reception methods;" et cetera. Unfortunately, such slogans refer to distinct instructional variables which often specify contradictory approaches to teaching if their range of appropriateness is not qualified and coordinated by at least an embryonic theory of instruction.

Educators appear to have little interest in theory construction despite the current interest in "model" building. Theory, too often, is used to justify instructional biases and the "laboratory" and/or "open-classroom" is considered only an instructional device, whereas theory construction is necessary to clarify the roles that different classroom structures have in educational research and practice. Also, a classroom laboratory should provide a context for research on problems relevant to specific aspects of the instructional process (variation of teaching strategies and techniques) and for discovering those psychological conditions critical for development of thinking in children.

However, in my opinion, Chesterton's remark is quite appropriate: "It is not that they can't see the solution--they can't see the problem." We have yet to identify the fundamental dimensions of educational and instructional problems facing the teacher. Key psychological principles for mathematics instruction and construction of a theory of instruction can be realized only after this first step has been achieved. A body of knowledge now exists in developmental-cognitive psychology that should have considerable utility for contributing to more refined theories of, and strategies for, instruction (i.e., for creating better school learning environments). An approach to the immediate task is to search for suggestions from developmental psychology that pose relevant problems for developing theories of instruction.

There is no paucity of choices of psychological "models" available from which to start the search. Berlyne, Bandura, Bruner, Scandura, Skinner, Suppes and, of course, Piaget have each proposed a set of ideas relevant to a theory of instruction requiring theoretical and empirical study. The selection of any one model brings with it many hidden presuppositions and is determined in no little part by one's own preconceived notions regarding human development, learning, and education. As a group, these models represent a virtual wonderland of exciting ideas. Each of us can understand Alice's dilemma better as we explore their fantasies (Suppes, 1972).

A theory of instruction must begin with an adequate theory of cognitive development and learning. No longer can we accept that statement as "obvious"--and go about the business of generating a multitude of methods based on unorganized intuitive rules based on inadequate knowledge of the process of cognitive development in children. All educators need to return to the beginning and ask, not "how do we teach?", but rather "how do children learn?"

Modern developmental psychology provides a necessary, but not sufficient, body of knowledge for identifying some of the fundamental issues, constraints, and facts associated with the process of generating a theory of learning and instruction. But, to imply and act as if psychology had become relevant to mathematics learning only AP (i.e., after Piaget) misses a fundamental point about the relation of the science of psychology to the science of education. It distorts the history of both. Piaget's theory of cognitive development should not be abandoned without clear understanding of why. The historical pattern in education and psychology seems to be one of enthusiastic adherence to a relatively novel theory--with disappointment and rejection following close behind. The absence of serious controversial issues underlying much of the current research in cognitive development increases my concern that much of what is valuable in Piaget's theory may be lost.

Natural Genetic Epistemology and Cognitive Development

Many psychologists, including myself, consider Piaget's clarification of the necessary bases of theory construction as important as his cognitive developmental theory *per se*. Implicitly and explicitly, Piaget was greatly influenced by advances in theoretical physics (Bridgman, 1927/1961) during the 1920's and 30's. The fundamental aspects of relativity theory that cannot be ignored in psychological theorizing are that a) conceptual judgments are always relative to the position of the observer and that b) analysis of knowledge acquisition requires a description of its operational basis (i.e., the mental operations of the individual associated with the construction and maintenance of consistent patterns (structure) of his continually transforming relations with his physical and social environments). Thus, Piaget is unique in that his emphasis on a constructivist theory of knowledge (Piaget, 1968, 1971a) is indissoluble from his interpretation of operationism. "Reality" is constructed, not imminent in mind, man or stimulus, and applies to the child and, contrary to some interpretations (e.g., Stevens, 1935), theorist alike.

The form of epistemology typical of American psychologists (cf. Mischel, 1971) has been naive realism. That orientation has been quite useful. Our epistemological preconceptions, whatever they may be, are part of our theoretical picture of the child. Kessen (1966) states the issue clearly: "The child who is confronted by a stable reality that can be described adequately in the language of contemporary physics, is a child very different from the one who is seen facing phenomenal disorder from which he must construct a coherent view of reality" (pp. 58-59).

Analysis of cognitive learning and development is always "biased" by the fact of a context of preconceived ideas of reality (i.e., Western culture) and a particular set of concepts or theory and selected observations. Piaget's approach to the analysis of the development of children's conception of space provides us with an excellent example (Piaget & Inhelder, 1956). The specification of a conception of space toward which the child will most likely develop, i.e., that conception held by most adults, is the critical first step. Observations and interpretations of the child's behavior are organized around the specifications inherent in that "endpoint" of development. This is not an example of bad science or of inappropriate procedures but rather illustrates that conclusions about the child's cognitive structure are often more a function of the construction of reality imposed on him by the scientists rather than of the reliability and generality of "objective" observations. Whether we are engaged in instructional practice, research, or theory building, there is, for each of us, a set of guiding propositions that constitutes a theory of learning and development. These "fantasies" or "freely invented ideas and concepts" provide a particular coherent view of the developing child and of the presumed determinants of learning.

Piaget's constructivistic epistemology, and his biological background, predisposed him toward an operational and structural analysis of the knowledge acquisition process (Piaget, 1967, 1968, 1970a, 1970b, 1971a). The essentials of his position require only brief review here. Knowledge is defined as invariance under transformation (a most familiar concept to mathematicians). The construction of invariances in organism-environment relations takes place through the operation of two complementary biological adaptation processes, both of which are under the control of the internal self-regulating mechanism of equilibration.

One of the two processes (assimilation) concerns the application of cognitive operational systems (structures) to the organization of sensory data. New data or events are incorporated into existing structures through both on-going physical and mental activity. Such events and the products of new experience can be incorporated into the cognitive structure only to the extent that they are consistent with existing functional structures. Accommodation is the complementary process whereby adaptation occurs by integration of existing structures with functional structures and/or by differentiation of new structures under confrontation with new experience.

Activities such as play, practical or symbolic, represent assimilative activity; whereas memory, in the sense of invoking past experience, and imitation are accommodative since only prior formed structures are transformed for a new use or application. Assimilation is an active constructive process by which the data from experience are transformed and integrated with an already generalized cognitive structure. Accommodative activity, on the other hand, is the process whereby modified existing structures or novel structures are brought to bear on newly assimilated sensory data.

Too often instructional theory and practice have emphasized assimilation (i.e., "play") or accommodation (i.e., imitation) activity and neglected the role of equilibration of these complementary processes for cognitive learning. However, generalization of Piaget's ideas to instructional theory and practice is not simple and straightforward. The special meanings attributed to "logic," the role of equilibration mechanism in constructing experiential data, and the distinction between operative and figurative thought are critical to such an endeavor.

Operational Structures and "Logic"

Piaget recently (1970b) elaborated his position that all human beings possess the same biological structures and functions that, in "exchange" with the common features of the natural world, generate mental (operational) structures and functions characteristic of each stage of development. Logical thought, in the Piagetian sense, is universal and of fundamental importance to an understanding of development and learning. But, whereas Chomsky maintains the human mind is "programmed" at birth with cognitive structures (i.e., mental representation of a universal grammar), Piaget accounts for the universality and stability of structures across cultures

(Goodnow, 1962; Goodnow & Bethon, 1966; Greenfield, 1966; Maccoby & Modiano, 1966; Piaget, 1967) in terms of the self-regulation mechanism of equilibration. Thus Piaget (1971b) proposes that the mind at any point in development is the unfinished product of continual self-construction. Logical processes are generative and not fixed. Structures are not preformed, but are self-regulatory, transformational systems with the functional factors in that construction being the processes of assimilation and accommodation.

Intelligence, the basis of knowledge acquisition, has two aspects: adaptation with the complementary process of assimilation-accomodation under the self-regulatory mechanism of equilibration and organization consisting of sets of mental operations that form the basis for maintaining invariance under transformation (i.e., knowledge). It follows from these considerations that there is an inherent logic to development. Operational systems consist of elements and laws of combination of those elements that form a "logical" closed system. These mental structures are observable in the actions of the organism in its environment. They are describable in terms of formal or logico-mathematical structures. Genetic psychological analyses of these structures are a necessary prerequisite to an understanding of thought processes since there is "no structure without genesis, no genesis without structure."

During the sensori-motor period of development, action structures of the individual are revealed in "practical" groups, i.e., the coordinated actions of the individual (Forman, 1973). During the preoperational period, the child constructs representations called figurative structures which do not have the operational property of reversibility. Piaget was able to identify operational structures with mathematical system properties in children between ages five and seven. The discovery of a resemblance between the structure of the mental action system (reasoning or thought) and mathematical structures (i.e., mathematical groups and lattices) had a profound effect on Piaget's thinking. Thought, it would appear, has the same or similar properties as mathematical group structures and both are governed by the same internal logic.[3]

The basic structuralistic approach of Piaget involves finding or creating logico-mathematical systems that describe the thought processes of an individual. Mathematical group and lattice theory are algebraic systems (Flavell, 1963, Piaget, 1957a, 1957b) which might well describe operational thought. Piaget found it necessary to generate a "grouping" model with additional properties (i.e., both group and lattice properties) to describe the concrete operational structures. Most importantly, the properties of these psychological groupings are not derived from the properties of things, but from patterns in relation to things. Thus, the elements of psychological groups are themselves transformations that characterize the individual's operations as he acts upon sense data.

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[3] Piaget never has tried to find a mathematical "logical model" to "fit" the observed facts of behavior; rather the mathematical aspect of Piaget's theory is unique in that he assumes, somewhat reminescent of Boole's "laws of thought," an identity between the inherent logic of thought processes and certain mathematical systems that have become formalized and "externalized" through inductive reasoning.

The revelations emerging from relativity theory require a constructivistic position with respect to the nature of knowledge. Understanding of knowledge acquisition requires a description and characterization of the mental operational systems applied to the data of experience. Piaget's emphasis on structural analysis thus is in terms of three traditions in modern intellectual thought: (a) the epistemological implications emerging from relativity theory; (b) the biologists' emphasis on development as the formation, differentiation and hierarchical integration of functional action structures; and (c) the mathematicians emphasis on formalized systems that permit description of these structures. The task of the developmental psychologist is to describe the nature of action structures of the child at each point in development and, as much as possible, to formalize those descriptions in terms of logico-mathematical terms.

The classical "conservation" tasks, if administered appropriately, form one basis for generating observation of coordinated actions that appear to reflect such mental operational structures. The available evidence appears to support the hypothesis that such operational (mental) structures "exist" both in terms of replicability of hypothesized developmental trends and in appropriately designed training studies (cf. Beilin, 1971b). At the same time, neither psychological nor educational researchers have yet devoted sufficient attention to the problems of the validity (i.e., internal consistency) of the grouping structures (Clarey, 1971; Green, Ford, & Flamer, 1971) nor to the role of such structures in learning (Berlyne, 1965; Bruce, 1971; Inhelder & Sinclair, 1969), beyond these few studies.

Only recently have educators, especially mathematics educators, become interested in Piaget's views of fundamental logico-mathematical relations, such as his ideas about the logical properties of number and space. Beilin (1971a) points out that philosophers of science generally have emphasized the desirability of isolating philosophical and logical issues from psychological matters. Psychologists, mathematicians, and logicians generally have maintained this position with respect to Piaget. However, a significant part of his psychological theory has mathematical and logical content which cannot be ignored by either psychological (Alonzo, 1970, Leskow & Smock, 1970) or mathematics-learning researchers. Mathematics education researchers rightly should be directed to the analysis of the logical and mathematical validity of Piaget's system and to the correspondence between the characteristics of the psycho-logic systems and those logic structures derived from purely mathematical analysis. Recent work from Steffe's laboratory (e.g., Johnson, D., 1975; Johnson M., 1975; Kidder, 1976) represents an excellent beginning in this direction.

Role of Experience and Equilibration

Experience is not the only critical factor in development according to Piaget. Merely being exposed to particular environmental situations is conducive neither to cognitive activity nor to developmental change. Children may or may not make discoveries in the course of play. Watching a laboratory experiment or conducting one may or may not help a child acquire a particular concept. Equilibration is the central factor in structural change whether that refers to "stage" or to concept learning. Equilibration is the process of the adaptational structure "controlling" itself (intrinsic regulation), balancing assimilatory and accommodatory processes, compensating for external and internal disturbances (internal or external to a particular structure), and making possible the development of more complex, hierarchically integrated operational structures. Rhythms, regulations, and operations are the three essential procedures of the self-regulation and self-conservation of structures. Anyone is free to see in this the "real" composition of structures, or to invert the order by considering the operative mechanisms as the source of origin. In any case, it is necessary to distinguish two levels of regulation. One level of the regulation remains internal to the already formed or nearly completed structure and, thus, constitutes a state leading to equilibrium. On the other level, such regulation participates in building up and integrating new structures.

Disequilibrium occurs as the child assimilates data from immediate experience into existing mental structures. As cognitive structures change to accommodate to the new experiential data, equilibrium is restored. The equilibration process is one of auto-regulation -- both of the transformations of data based on existing cognitive structures and of changes through accommodation. Thus, the child must be exposed to situations that are likely to "engage" the functional structures. He must be involved in a personal striving to understand or "accept" the task as a "problem."

A basic question for instructional theory and practice is: What are the processes and conditions that motivate and insure engagement or acceptance of the problem task by the child? The source of "interest" that promotes striving for problem solution is contingent on assimilative-accommodative activity, but the specifics remain unclarified in Piaget's theory (cf. Mischel, 1971). Within a structuralist framework -- if a structure exists, it must function -- multiple cognitive structures provide a dimension of openness that make probable continual sources of disequilibrium from interaction of the internal operational and/or figurative structures activated, as well as by exchanges involving novel experience. Despite lack of specifications, Piaget is quite explicit on his general position:

> It is not necessary for us to have recourse to separate factors of motivation in order to explain learning; not because they don't intervene. . . but because they are included from the start in the concept of assimilation. . .to say that the subject is interested in a certain result or object thus means that he assimilates it or anticipates an assimilation and to say that he needs it means that he possesses schemas requiring its utilization. (Piaget, 1959, p. 86)

In passing, it might be noted that natural or life-like contexts seem to provide excellent situations for promoting cognitive change. Unfortunately, too little empirical investigation has been oriented to questions of the natural environmental determinants of curiosity of children at various stages of development and with different experiential backgrounds. What do children recognize as problematic? What kinds of incongruities are sufficient to motivate change in concepts and/or beliefs?

Cognitive conflict, or the awareness of a momentary disequilibrium, generates a need to establish equilibrium between the existing schemas and/or novel information. This condition is the motivation for cognitive activities. Both application of an existing schema, and the elaboration of new ones in the course of development, stem simply from the overriding need to make "sense" of present problems by fitting them coherently into schemas "learned" in the course of solving prior problems.

The notion that disturbances introduced into the child's systems of prior schemas lead to the adoption of a strategy for information processing is the fundamental difference between the equilibration and associationistic theories of learning (Piaget, 1957b). For associationistic theories of learning, "what is learned" depends on what is given from the outside (copy theory), and the motive that facilitates learning is an inner state of some sort or other. Equilibration theory holds that learning is subservient to development; what is learned depends on what the learner can take from the given by means of the cognitive structures available to him. Further, cognitive disequilibrium is the functional need that motivates learning. Questions or felt lacunae arising from attempts to apply schemas to a "given" situation are disequilibrium. The child will take interest in what generates cognitive conflict or in what is conceived as an anomaly. If the task demands are so novel as to be unassimilable or so obvious as to require no mental work, the child will not be motivated.

After the period of sensori-motor development, equilibration becomes a process of compensating for "virtual" rather than actual disturbances. At the operational level, intrusions "can be imagined and anticipated by the subject in the form of the direct operations of the system--the compensatory activities also will consist of imagining and anticipating

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[*] Translation by E. von Glaserfeld

the transformations to an inverse sense" (Piaget, 1957b, p. 93).* Further, there need be no external intrusions in order for the equilibration process to be activated. For example, the acquisition of conservation concepts is, in Piaget's view, "not supported by anything from the point of view of possible measurement or perception -- it is enforced by logical structuring much more than by experience" (Piaget, 1957b, p. 103).* It is the internal factors of coherence -- the deductive activity of the subject himself that is primary. Equilibration is a response to internal conflict between conceptual schemas rather than a direct response to the character of environmental structure factors.

Operative and Figurative Thought

A considerable amount of confusion concerning Piagetian theory and its implication for both research and instructional practice derives from a failure to consider the figurative and operative aspect of intellectual functioning. In general psychological terms, the distinction is between the selection, storage and retrieval versus the coordination and transformation of information (Inhelder, Bovet, & Sinclair, 1967). More specifically, the development of any sequence of psychological stages, a la Piaget, consists of an interactive process of equilibrating functional structures of the organism with sensory event-structures of the perceived environment. Piaget (1970a) analyzes "experience" into the two components: "physical" and "logico-mathematical." This distinction between a physical and logico-mathematical experience is essential to the understanding of the growth of knowledge. Knowledge based on physical experience alone is knowledge of static states of affairs; if a child reasons incorrectly in a physical experience, it is easy to demonstrate that he is wrong. While knowledge emerging from logico-mathematical experience is knowledge of transformation of states and quite another matter. If a child reasons incorrectly in a logico-mathematical experience it is difficult, if not impossible, to demonstrate convincingly, or even to get the child to accept verbal explanation of the "correct" answer. For example, if a child fails to align the two endpoints when comparing length of sticks, it is easy to correct the mistake. If, however, he fails to display transitive reasoning in a task, one or two examples are not likely to "teach" him the concept.

Physical experiences provide for the construction of the invariants relevant to the properties of states of objects (figurative processes) through exchange with objects involving sensory mechanisms. For example, one may touch something and it is hard, cold, hot, soft, supple, et cetera.

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[*] Translation by E. von Glaserfeld

Or one may see something -- an object is red, a diamond cutting glass, the shape of a banana, et cetera. The course of logico-mathematical experience is assumed to be abstractions (operative knowledge) from coordination of actions vis-a-vis representations of "objects," or transformations of the "states" associated with series of discrete physical experiences. The critical difference is that logico-mathematical knowledge demands that a pair (or set) of physical objects not be defined by the temporal-spatial (perceptual) similarities, but rather by invariant relations among or between objects.

Figurative and operational processes represent two types of functional structures necessary to account for knowledge acquisition. Figurations are defined as those action schemata that apprehend, extract and/or reproduce aspects of the prior structured or organized physical and social environment. Such action schemata include components of perception, speech, imagery, and memory. Figurations and associated acts are based on physical, as contrasted to logico-mathematical, experience and consititute the "empirical" world. Empirical truth is no more than the "representation of past experience in memory."

Operations do not derive from abstractions from objects and specific events; rather, operational knowledge is derived by abstractions from coordinated actions relevant to those events. Thus, operations are those action schemata that construct "logical" transformations of "states." Such "logical" systems of transformations operate either upon representations of events, or on the cognitive system's own logical operations, i.e., reflexive operations.

Figurative and operative structures are two parallel streams with their genetic or developmental origins in the same source (Piaget, 1967, 1968, 1970a, 1970b; Piaget & Inhelder, 1971) -- the sensori-motor structures. Logical (operational) structures are not immediately generated by the figurative schemata alone (i.e., not from perception, memory, et cetera). Reciprocally, figurative structures do not derive from operative schemas but from the representations of past states of events derived from physical experience. Most importantly, figurative structures do not derive from each other, but have unique bases in sensori-motor schema. Imagery, for example, is a derivative of deferred sensori-motor imitation (Piaget, 1951, 1952; Piaget & Inhelder, 1971) and not perception.

The postulation of these quite different levels of functional structures is one of the cornerstones of Piaget's theory of knowledge acquisition and cognitive development (cf. Furth, 1969). The source and function of each structure is theoretically distinct. Operative structures derive from abstraction from coordinated actions. Figurative structures derive from sensori-motor and perceptual activity. Operative structures produce "logical" transformations (conservation of invariants) whereas figurative structures reproduce sensory perceptual consequences of externalized or "environmental" configurations. The variant operative structures of the intuitive, the concrete, and the formal levels form the discontinuous sequence of stages of cognitive development. Figurative structures are static and depend directly upon the data of experience (sensory-perceptual consequences of stimulation). Piaget makes the fundamental assumption that all knowledge acquisition activity is constructive, but the construction of figurative representations is quite a distinct process from the constructive activity at the operative level.

Logically, there are three possible relations between the figurative and operational structures (Langer, 1969). First, they may be unrelated. If so, as mentally segregated functional structures, they do not set limits on the functioning and development of each other. Second, psychological phenomena might be reduced to one of the types of structures. Langer (1969) suggests that subjective idealists try to reduce psychological phenomena to assimilatory operations. There are many theorists who try to reduce all mental phenomena to accommodatory figurations and the naive realists propose that all knowledge is figurative (e.g., perception is knowledge) (Garner, 1962; Michotte, 1963). Third, Piaget proposes there is a partial communication between figurative and operative structures within the constraints of the assimilation and accommodation processes. The relations and the potential from of interaction of the components of adaptation and organization discussed above are schematically presented in Figure 1.

Langer (1969) has examined the organizational and developmental (i.e., transitional) impact of accommodatory figurations on assimilatory operations (B, Figure 1). This is equivalent to asking how does the child mentally extract and/or represent empirical information about physical and social objects and the consequences of that empirical activity for the construction of logical concepts. Imitation of an observed event, comparison of one's predictions with the perceived outcome of a physical manipulation, comparison of an observation or appearance (i.e., immediate experience) with the way things have been constructed and externalized, represent different modes of introducing internal conflict and cognitive-structural change. Generally, Langer's findings are confirmatory, but not definitive with respect to the Piagetian hypotheses. In any case, if the development of each type of functional structure has implications for, but not direct causal effects upon, the structure and development of the other, current paradigms for the study of learning mathematical concepts will require considerable modification. The work of the Geneva group mentioned earlier, concerning for example, memory (A, Figure 1) and Langer's (1969) analysis of the impact of accommodatory figurations (i.e., imitation, etc.) on assimilatory operations represent beginnings in this direction.

Analysis of learning, in the context of Piagetian theory, poses requirements for more detailed empirical analysis than has been generally recognized. On the one hand, researchers attempting to assimilate Piaget to

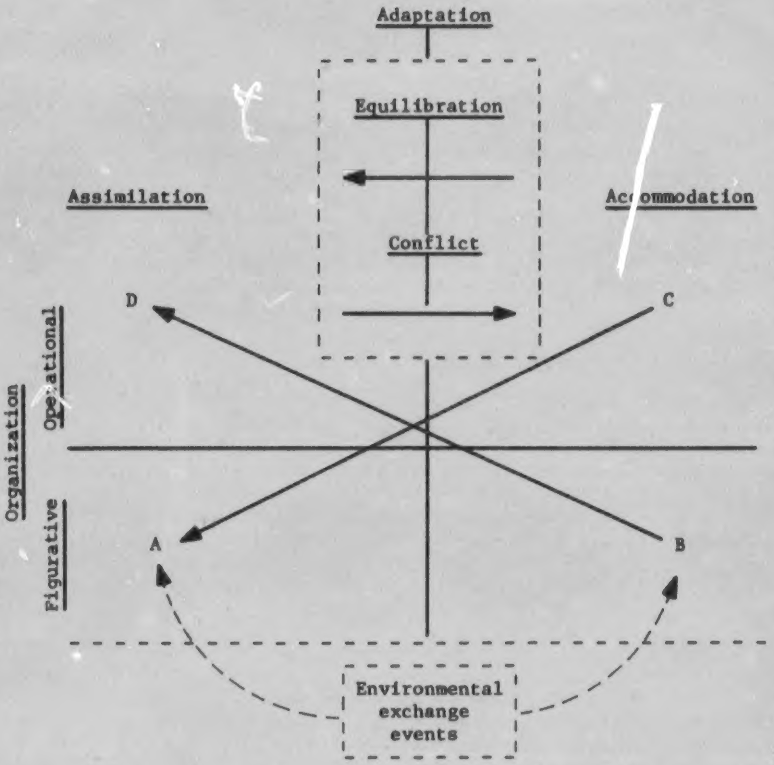
Adaptation
Equilibration
Assimilation
Accommodation
Conflict
D
C
Organization
Operational
Figurative
A
B
Environmental exchange events

Figure 1. Relations of two invariant processes of adaptation and two types of cognitive structures.

their own concetual structures concentrate on experimental procedures whereby the subject is required only to remember event contingencies or similar figurative structures (e.g., response-reward associations or "a" follows "b", follows "c"; Bruner, Rose, & Greenfield, 1966). Such procedures certainly produce change in "behavior" (e.g., Bever, Mehler, & Epstein, 1968; Gelman, 1969; Mehler & Bruner, 1967), however, failure on transfer tasks and a lack of persistence of task solution over time indicate that a figurative process underlies the change in performance. On the other hand, the accommodators (i.e., those more favorable toward Piaget's theory) often fail to generate experimental paradigms that adequately differentiate between the figurative and operational knowledge (Wallach & Sprott, 1964) or assume that "external disparity" (apperance vs. "reality") is sufficient to establish disequilibrium or conflict between logical necessity derived from the operational structures and perceptual pregnance (cf. Bruner, 1966). Situations designed to establish disparity between the child's predictive judgment of the outcome of a transformation and his observation of the actual outcome may, in fact, generate little or no cognitive conflict. A most parsimonious explanation of many "negative" findings in training studies is that such disparity belongs to the experimenter's "reality" and is external to the child's own logical operational system.

Implications for Learning and Instruction

In some form or other, the goals of American educators have always been stated in terms of "optimizing" the intellectual, social, or alternative aspects of development of individual children. Whatever such goals imply, the educational and instructional processes must be based upon an understanding of the nature of psychological development of children. Whether we want to produce individuals who will strive to maintain the status quo, individuals who desire and accept change, people content to be technologists, or problem solvers, it is necessary to understand the basic processes of child development and the conditions that permit "quality control of the product."

The issue is important because science can only yield "what is" and not what "ought to be." We are fortunate, in one sense, that the sciences of psychology and pedagogy are young and imperfect. The proposed models and methods for educating young children are no less imperfect and are influenced as strongly by current social thought and individual philosophical biases as by an understanding of the laws of psychological development. Such a state of affairs, while producing wasted efforts, spurious claims, and more rediscoveries than discoveries, may hopefully provide time for the development of articulated sets of societal goals for education.

The best that can be hoped for, under the current conditions of our knowledge, is development of preliminary "models" for instruction. Such models can provide, at least, a schematic set of principles and guidelines for constructing a learning environment consistent with the admittedly inadequate theories and knowledge of psychological growth. However, we should try not to violate recent advances in theory and known laws of child development.

Piaget, until recently (1971b), declined to generalize his theory to specifics for educational practice. His theory of knowledge acquisition has contributed to clarification and integration of a set of propositions about psychological development, many of which have a long history in child psychology and education. If we accept his theory of cognitive development, several deductions concerning the construction of "optimum" environments can be generated. A modest attempt in this direction has been made at the University of Georgia for the Follow Through Program (Smock, 1969). Though the basic propositions of the model are not inconsistent with Piaget's thinking about knowledge acquisition, the interpretation is that of the modeler. It is influenced, therefore, by numerous sources of bias, misunderstanding, and distortions that are inevitable under conditions where abstract theoretical concepts are not represented in unequivocal abstract or logico-mathematical terms.

We start with the general proposition that the child is not a passive recipient of stimulation, nor can external reinforcement be considered a primary factor in learning and behavioral change. Further, the introduction of "mediation responses" (verbal or otherwise) is not able to account for the complexities of observed changes in behavioral organization during the course of psychological growth during childhood. Many psychological theorists have adopted, in one form or another, the idea that human organisms actively respond to their environment and that the patterning of these responses reflects a "plan" or "set of cognitive operations." In other words, the child "interprets" environment input, and the interpretations are controlled by his capabilities for generating rule systems for coordinating and transforming the input to "match" a scheme, plan, or a mental operational structure. Analysis of the "rule systems" characterizing cognitive development, thinking, and learning, requires specifications of the properties of, and antecedent conditions for, selection and structuring of the consequences of environmental events (mental representation/figurative knowledge) and of the mental actions (operative knowledge) necessary for coordination and transformation of those representations. The study of the development of rule systems defined as such is coincident with the systematic investigation of the "inherent logic" of development of operative and figurative thought processes.

Intelligence, first of all, is considered no more, and no less, than biological adaptation. Adaptation at any level of complexity reflects "intelligent" activity. "Knowledge" consists of two types of functional structures (figurative and operative) that give rise to invariants in organism-environment relations. These invariants are derived from abstractions from objects (physical experience) in the first case, and from coordinated actions (logico-mathematical experiences) in the second. Intelligence, then, refers to both types of cognitive learning and is defined in terms of functions (i.e., thinking or reasoning) rather than content (i.e., words, verbal responses, associations, et cetera). Analysis of conditions for cognitive learning and development must begin with the identification of components of behavioral organization (structure) that reflect particular coordinated action-modes of the child as he is confronted with changing intrinsic (maturation and prior cognitive acquisitions) and extrinsic (physical and socio-linguistic) factors.

Cognitive structures or systems of coordinated (mental) actions proceed through invariant stages of structural change with ontogenetic development. The successive differentation and hierarchical integration of these cognitive structures permit the individual to cope with increasingly complex social and physical "realities." The process of cognitive development involves the changing characteristics of transformational rule systems (virtual and/or cognitive operations) characterizing the child's mode of adaptation. Neither the maturational structure of the organism nor the "teaching" structure of the environment is the sole source of reorganization; rather, it is the structure of the interaction (exchange events) between the child and the perceived environment that provide the basis for intellectual development.

Optimal conditions for structural organization and reorganization require: (a) an optimal degree of discrepancy between perceived environmental/demand structures (i.e., perceptual activity, images, memories) and cognitive operational structures; and (b) social-learning conditions that demand "spontaneous" or "constructive" activity by the child.

Several implications for the construction of theoretically appropriate learning environments are implied in these general principles. First, structural change depends upon experience but not in a way that traditional learning theorists conceive of experience or learning interpreted as pairing of specific objects and responses, direct instructions, modeling, et cetera. Rather, the functional genetic view holds that the cognitive capacities determine the effectiveness of training. For example, ability to solve class inclusion problems implies that the child already has the requisite single and multiple classification operational system for classes (i.e., combination, reversibility, et cetera) in addition to appropriate information selection, storage, and retrievel abilities. At the same time, while experience is necessary for developmental progress, and appropriate enrichment of the environment can accelerate such development, experience cannot change the sequence, structuring, or emergence of action modes in the process of developmental change. In other words, organization of experience is not provided solely by the environment nor solely by the structures intrinsic to the child.

Second, the structure of the learning environment must be considered relative to two frames of reference: in terms of the operational systems controlling the child's interpretation of "environmental" events and the content to be learned. Operational systems are expressed behaviorally in the coordinated actions of the child confronted with changes in his physical and social world. For example, the mental operations of associativity or reversibility are inferred from the manner in which the child attempts to solve problems involving regular environmental contingencies or causality understanding of spatial relations, arithmetic and the other substantive areas (such as science or mathematics). Each must be analyzed in terms of their own logical sequence and commonalities with other content areas. Content concepts in the physical sciences, languages, and mathematics, for example, may have an inherent sequence and structure. Thus, certain concepts may be necessary precursors to subsequent understanding of higher order concepts. Optimal educational conditions require, then, thorough understanding of the psychological-cognitive capacities of the child as well as the sequential structuring of concepts within a particular curriculum area.

Third, the striving for equilibration between assimilatory and accommodatory processes under both intrinsic and/or extrinsic pressures underlies the adaptive process. *Optimal conditions for structural reorganization*, learning in the broad sense, *require disequilibration*. This condition is met when there is an appropriate "mismatch" between the cognitive capacities of the child and the conceptual demand level of the learning task. Too little or too much "pressure" may result in over-assimilation or over-accommodation respectively, and not promote cognitive-developmental change.

Fourth, facilitation of learning requires analysis of two levels of cognitive functioning -- figurative and operative processes. The first is most emphasized by those theorists, particularly behaviorists, recommending a direct tuition approach to instruction. The operational theory of intellectual development does not deny the value of "provoked" learning (i.e., through imitation, algorithms). Rather, such learnings are considered limited because of lack of generalization or transfer to new situations and because the basic intellectual processes concerned with problem solving and reasoning are not significantly affected.

While there is some doubt that much *acceleration* of structural reorganization is possible through environmental enrichment, early childhood education should provide opportunities for utilization of relevant cognitive operational structures. Generalization of conceptual learning across content areas rather than the building of specific knowledge and skills (e.g., a large vocabulary) should be emphasized since the latter cannot directly accelerate operational system change and may, in fact, retard development of these "deeper" competence structures.

In any case, the *nature* and *variety* of the child's "exchanges" with the environment need to be considered in educational planning. The *nature* of the interaction refers to the *relative* emphasis on ontogenesis (self-directed) as contrasted to exogenesis (environmental or teacher-directed) structure of the learning environment. The functional genetic position can best be summarized in the old adage -- "you can lead a horse to water, but you can't make him drink - unless you feed him salt." Thus, the task of the teacher is to engineer an educational environment consisting of curriculum materials, social interactions, and directed activities that provide appropriate "salt" for each child. Sequentially structured curricula should be designed to provide an optimal degree of structure and conceptual level to permit an appropriate balance of assimilatory and accommodatory activity.

The variety of interaction or enrichment refers to the types of structured curriculum content relevant to the child's physical, social, and symbolic experiences. The interlocking nature of substantive curriculum areas makes it possible to provide a variety of experiences relevant to acquisition of the cognitive "products" that provide representation of the environment, such as memories, vocabulary, or symbols and, at the same time, to facilitate the development of coordinated rule systems associated with cognitive operational development. For example, analysis of the visual environment (attention or observational skills) as well as cognitive operational structures (e.g., conservation of area) can be emphasized in science, social studies, mathematics and art.

The engineering of an educational or "learning environment" based on the preceding consideration necessarily involves the development of specifications of: (a) the child's cognitive developmental level; (b) the physical structures, including curriculum materials; and (c) the social or interpersonal structures. The organization of these "elements" should be such that equilibration, between different cognitive systems and/or between intrinsic functional structures and "environmental" structures, is achieved. Thus, sequentially structures sets of curriculum materials and of social interaction situations are necessary to provide the "pressure" necessary for learning and adaptation. A variety of specific learning environments needs to be available to maximize the probability of each child's finding activities that attract or "trap" him into interacting with the physical and social environment at both the behavioral and symbolic levels of language and mathematics in creative and spontaneous ways be it through art, role playing or music. Finally, the physical and social environments should be arranged so that considerable freedom of movement, within the structure of a variety of contents, is possible, i.e., "a modified open-structure classroom." A careful balance between relatively high and low structured learning situations and between group and individual learning activities should be maintained.

The Mathemagenic Activities Program, a model developed in the context of enriching the educational environments of economically deprived children,[4] is based on three explicit principles derived from the considerations discussed above. Specifically, the MAP principles of change --

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[4] Mathemagenic Activities Program: A Model for Early Childhood Education, prepared by C. D. Smock (A preliminary statement of the conceptual basis for the Mathemagenic Activities Program for the Follow Through Program appeared in Terminal Report (1970)).

whether the target for change is the individual organism (child) or a complex social system (e.g., Local Education Authority)--are based on the above assumptions concerning the role of experience in learning and development. First, the source of motivation to change is provided by a discrepancy (disequilibration) between different conceptual systems (ideas) and/or between previously acquired conceptual systems and environmental task demands. Thus, an appropriate mismatch (M) is necessary to generate exploratory activities and insure the individual has the prerequisite conceptual basis for learning higher order concepts.

Second, since coordinated actions (practical and mental) are the bases for knowledge acquisition, the learning environment must be structured so that specific task demands include appropriate practical, perceptual, and mental activity (A).

Third, the learning environment must include provisions for personal self-regulatory (P) constructions. Knowledge acquisition involves construction of invariants from properties of objects (physical experience) and from the child's actions on objects (logico-mathematical experiences). Optimal conditions for facilitating new "constructions" (concept learning) involve a balance between tasks that are highly structured (in which the child merely "copies" or imitates the correct solution) and tasks that permit the child to generalize and discover new applications of his concepts. Practically, self-regulation implies a variety of task options available to the child; the number of options may well vary with the nature of the task and many other factors. MAP proposes, however, that options--in terms of level of task difficulty, mode of learning, and choice of activity--are necessary ingredients of developmental change, whether the target be a child, a teacher, or an educational system.

The implied educational model requires significant changes in the teachers' role definition and teaching strategies and tactics. The need for sensitivity to the child's capabilities, and the structuring of learning situations that promote self-regulated, "constructive" knowledge acquisition, together with thorough acquaintance with available technological aids, require an "educational engineer" in the best sense of that term.

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Problem Solving in a Model for Early Mathematics Learning

Doyal Nelson

University of Alberta

A number of mathematics programs for young children have been developed in recent years which are modelled after various theories or hypotheses about how children learn. The Montessori program (Orem, 1971), based on a "prepared environment" around which mathematical experiments are arranged, is probably the best known. More recently there is the highly structured DISTAR Program of Englemann and Carnine (Englemann, 1969) which leans heavily on a task analysis of mathematical skills provided by Gagné (1972) and his hierarchical notion of how experiences should be arranged. Kamii (1971) is attempting to arrange an entire program of preschool education around the processes of development found in Piaget's work. A good deal of research is being done to evaluate these and other developing programs (Suydam, 1974). However, another alternative which has been widely discussed but which has not been adequately investigated is what might be called the problem solving approach.

It would appear that mathematics instruction in the early years should be aimed primarily at helping children solve problems associated with situations and events which occur in their lives from day to day. Such problem solving situations for young children could not be the usual kind foun in mathematics programs where the solver is expected to find some mathematical expression to fit reality. The system of symbols available to young children is too incomplete to process problems of any consequence this way. However, it should be possible to get children involved in solving problems which require the use of a variety of mathematical processes but which require the use of no symbols at all. There is evidence that such processes are available to young children and it might be assumed that their application to problems in the real world could provide a sound basis for mathematical abstraction.

The purpose of mathematics instruction at say ages three to eight could be best served when the child gains new control over, and a deeper understanding of, some aspects of reality and can in turn transform or reorganize it in ways that more nearly suit his own ends. Such control would be manifested when he learns to share a number of toys or candies fairly and equitably; when he can draw a map to show how to move from one point to another on a plane; when he can arrange a row of markers in a straight line; when he can construct a series of towers each higher than the one before, and so on.

It might be assumed that any progress toward abstraction and mathematical understanding would require that the child have wide experience in dealing with problems of this kind. It would appear in any case that the child comes in contact at a very early age with a surprising number of processes which are, and which continue to be, of importance mathematically. If one can imagine a child playing with a set of discrete objects he will, depending on the nature of the objects, separate them, combine them, pile them, group them, partition them, compare them, order them, classify them, label them, and the like. When considering the spatial aspects of a three dimensional solid, for example, he may represent it, project it, dismantle it, turn it, bounce it, copy it, and so on. Many such processes find expression in mathematical symbols in the course of the child's early school experience and later mathematical experience would certainly require it. But is there any guarantee that the child has a good grasp of an idea or process as it applies to reality before he is required to represent it symbolically?

In the present situation, it is doubtful that this question could be answered in the affirmative. In the first place, the way most mathematics programs are organized for early grades it is necessary for the teacher to introduce the symbols of mathematics whether the child fully understands what the symbols mean or not. Second, if the teacher wanted to arrange experiences to guarantee basic understanding before symbols are introduced, there is not enough specific research evidence upon which to base a comprehensive program of such experience.

To illustrate some of the difficulties associated with establishing a problem solving program for the purposes outlined, consider a single example. Suppose one wished to arrange a series of experiences to guarantee that a young child understood measurement division before he was required to do exercises involving the division algorithm. In its simplest manifestation one would expect that there would be a set of objects and directions given to the child to find the number of equivalent subsets each with a specific number of objects. Presumably the objects would be movable, would fit nicely in the child's preceptual field, and would all be the same. What would the child's response be if the number of objects in the subsets was a factor of the number in the original set? How would the child respond if there was a remainder? What if the objects were not all in the same class? How would the child behave if the objects were to be grouped in another location? Would the way the directions are given make a difference? Can very young children cope with the problem? Do they behave differently in the face of distracting elements of the problem than do older children? How real does one make the problem? What kind of variables influence the child's problem solving behavior in this situation and how should they be controlled? If one wanted to generalize division to include partitioning, the questions posed above (and possibly some others) would have to be asked again. Answers to such questions would have to be available on all aspects of a mathematics program if one were to include problem solving in a model for mathematics learning. It is important that attempts be made to seek these answers.

Certain features are characteristic of any model for mathematics learning. Assumptions have to be made, for example, about how children in a particular age range learn mathematics; mathematical content has to be specified; attention must be given to the physical and social situations in which the content is to be learned; and finally, some empirical information must be available to permit interpretation of any variation in behavior exhibited by children in the learning situation.

If problem solving is to be included in a model for early mathematics learning, a number of steps must first be taken. First, criteria need to be established which will serve as guidelines in designing the problems. The works of Piaget, Bruner, Dienes and others would of course, provide initial guidance in designing such guidelines. Once the criteria were established, their potential as guidelines for creating good problem situations would have to be tested. Such tests should suggest ways of refining the criteria to make them increasingly effective. Since little is known about how children at various ages behave when presented with such problems some procedure would have to be developed to collect such behavioral information. The behavioral data should serve as a means of making further refinements to the criteria. The research model for studying probem solving behavior of young children is shown below.

A Model for Studying Problem Solving Behavior In Early Childhood

The purpose of this paper is to describe the progress that has been made so far on the research model.* The preliminary research does not

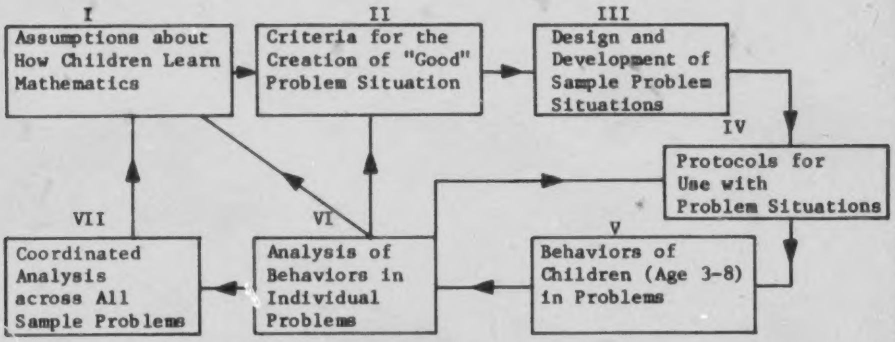
I
Assumptions about How Children Learn Mathematics
II
Criteria for the Creation of "Good" Problem Situation
III
Design and Development of Sample Problem Situations
IV
Protocols for Use with Problem Situations
V
Behaviors of Children (Age 3-8) in Problems
VI
Analysis of Behaviors in Individual Problems
VII
Coordinated Analysis across All Sample Problems

Figure 1. The main features of the research model.

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*The project designed to do this is being supported by Canada Council, a federal research agency which supports basic research in social science and the humanities: Director, Doyal Nelson; Co-Investigator, Daiyo Sawada.

address itself directly to the matter of including problem solving in a model for mathematics learning. It is, however, an example of the kind of research that needs to be conducted before adequate models for learning can be designed.

The Assumptions

Although certain assumptions, both about the role of problem solving in mathematics learning and about mathematics learning in general, are implicit in the opening discussion, the primary assumptions on which the project is based are as follows:

1. Significant problem solving situations can be devised for young children and their solution does not depend on a complex knowledge of mathematical symbols or expressions.

2. Mathematical abstraction and understanding is based largely on solving problems involving real objects and events.

3. Action is the single most important aspect of such problem solving situations.

4. Observable behavior of young children in problem situations can reveal important information about their mathematical understanding.

5. Interpretation of such behavior can provide some guidance in designing problem solving experiences for the promotion of deeper and clearer understanding of various aspects of mathematics on the part of the child.

6. Children learn in the presence of "noise" or distracting elements. Such "noise" should appear in the problem situations.

The Criteria

Some years ago, a colleague, Dr. Joan Kirkpatrick, and the investigators set about to construct a model which could be used to create "good" problems for young children. We wanted to be reasonably sure that if the criteria stated in the model were adhered to, that the resulting problem or problem situation would stimulate problem solving activity on the part of the child. The criteria which finally emerged are as follows:

1. The problem should be of significance mathematically. It is for the potential of the situation as a vehicle for the development of mathematical ideas that a particular problem situation or family of situations is chosen above all others.

2. The situation in which the problem occurs should involve real objects or obvious simulations of real objects. The main consideration here is that it be comprehensible to the child and easily related to his world of reality.

3. The problem situation should capture the interest of the child either because of the nature of the materials, the situation itself, the changes the child can impose on the materials, or because of some combination of these factors.

4. The problem should require the child to make moves or transformations or modifications with or in the materials. It is difficult to overemphasize the role of action in early childhood learning. Most of the mathematical models we are interested in at this level are what might be called "action models."

5. If possible, problems should be chosen which offer opportunities for different levels of solution. If the child can move immediately from the problem situation to an expression of its mathematical structure it is not a problem.

6. Whatever situation is chosen as the vehicle for the problems, it should be possible to create other situations which have the same mathematical structure. That is, the same problem should have many physical embodiments. It may not be possible for a child to generalize a solution to a certain structure of problem until the problem has come up in a variety of situations. Abstraction and understanding is probably facilitated when the child sees more than one physical situation embodying a particular mathematical idea or process.

7. The child should be convinced that the problem can be solved and should be able to show when he thinks he has a solution for it. If the child is somehow required to react with or transform materials used in problem situations, it is usually easy to determine whether the problem meets the criteria or not.

The model is still crude, but its application has been reasonably effective in providing direction for the creation of productive problem situations to study the behavior of young children. It is necessary at this point to introduce the following three definitions:

a. problem situation - all aspects of apparatus designed according to the criteria listed above;

b. problem - the apparatus and the accompanying verbal statement or demonstration designed to stimulate some reaction on the part of the child; and

c. equivalent problem solving situation - a situation involving the same general mathematical process or idea as another but not designed necessarily to stimulate the same problem solving behavior on the part of the child.

Creating the Problem Situations and the Problems

In applying the criteria to the creation of sample problem situations it was decided to limit their number to six but to develop, for each of these six, one equivalent situation. The general mathematical areas involved were:

1. division - measurement and partitive, no remainder.
2. co-ordinated reference systems - two dimensional and three dimensional.
3. sequences - alternating with two or more elements.
4. reflections on a plane.
5. factors - prime and composite numbers.
6. geometric representation - three dimensional in two dimensions.

Appendix A contains a complete description of the division problems and their protocols. Appendix B contains a brief description of all the other problems.

Sampling Procedures and Recording Behavior

The purpose from this point on was to determine if the problems would stimulate interpretable behavior on the part of young children. The age range of particular interest was three years to eight years. In that range many of the processes of mathematics are encountered for the first time and at the upper end of the range many mathematical situations and processes are being represented in mathematical symbols. Interview protocols were devised to permit a child to present any kind of response to a problem from the purely physical to the purely symbolic.

For the preliminary work, a sample of fifteen children from each age level three to eight was selected. Children were volunteered by their parents and came from an area served by five schools in the vicinity of the University of Alberta. It is recognized that voluntary samples may show specific kinds of bias, but this shortcoming was not considered to be serious in a preliminary survey where very little is known about the behaviors that might occur. In any case it would be very difficult, if not impossible, to obtain a truly representative sample of children as young as three years of age. No biographical data except age was collected in any systematic manner.

The laboratory was set up with a videotape recorder, two cameras, and a monitor. The child came into the room with his parent. An interviewer sat at a table at which a problem situation was displayed. The child was asked to sit in a chair beside the interviewer and the problem was presented according to protocol. While the child solved the problem the two video cameras (and sometimes a super 8 movie camera) were trained on him and recorded whatever he did. Split-image capability permitted two opposing views to be recorded at the same time. Counting the parent, laboratory assistants, technicians, and interviewers, there were usually about eight adults in the room. As soon as one problem was completed, assistants removed the apparatus and presented another set. Six separate problem situations were presented to each child according to a strict schedule. The schedule provided that ten children at each age level did each problem and that five of these did the equivalent as well. The decision as to which would be the problem and which the equivalent was made randomly before the data-gathering began.

To begin with there was some fear that the child might refuse to react because of the rather overwhelming laboratory setup. Most children however, were apparently not influenced by this. In fact, all children but one completed all six problems. That one, a three year old, did a single problem and would not go on. It was appraent that it was not the laboratory situation that upset him but rather an unpleasant experience on the way to the laboratory.

Protocols were followed closely unless a child either failed to make any response or continued to make responses that were unproductive. The interviewer in these cases was permitted to intervene. Children took from twenty minutes to one hour to do the six problems. (It should be noted that all available children will come back in the summer of 1975 after one year and do six more problems. This longitudinal aspect should provide a check on the validity of interpretations.)

Analysis of Behaviors

There are a number of steps which must be taken in analyzing taped data. In the first place, some decision has to be made about which behaviors are significant enough to be included in the analysis. Second, sufficient time must be spent viewing sequences to be sure that all significant behaviors are considered. Third, a coding system has to be devised which will convert the data into a form which can be readily analyzed. Finally, adequate reliability checks must be devised so that one is reasonably sure that all significant behaviors have been considered.

Although coding has been completed, analysis of the coded data has not proceeded far enough to permit a complete report. In order to illustrate the general form of analysis and to indicate the directions these analyses might take, preliminary work has been done on those problems involving measurement and partitive division. A report of these resulting analyses is included in this paper.

Results for Cargo Groups and Animal Groups - Measurement and Partitive Division (see Appendix A for details of the problem situations and the corresponding interview procotol)

These problems were designed to find how children would behave in problems involving measurement and partitive division. In the cargo groups, the child was involved in finding the number of groups of three cars in fifteen (measurement division). Then he was involved in finding how many cars would be at each of three houses if the fifteen cars were distributed among the houses so that there were the same number at each house (partitive division). The cars were all from a single set of small plastic models which represented a variety of makes in a variety of colors.

It was assumed that the older children might merely count the cars and then divide the number by three to get the result. They might on the other hand group the available cars in threes for the measurement situation or in three equal groups for the partitive situation. It was expected that younger children would have no systematic means of attacking the problem.

In the animal groups, the child was asked to tell how many cages would have to be built so that there would be five animals in each cage. The child was given an assortment of twenty plastic animals. The partitive question asked the child how many animals would be in each of three cages if each cage were to have the same number of animals. Again he had available an assortment of plastic animals, but there were eighteen animals this time. Animals were selected so that a classification of them based on what kind belong together in a cage would not give the solution. The most incongruous was a single lion which some children might be reluctant to introduce into a cage with other animals. This problem, too, could have been solved with minimal physical movement of the animals.

Certain distracting elements were built into each of the problems. In the cargo groups, for example, the ferry was to hold three cars, but there was obviously room for four. Also when the child unloaded the cars there was nothing to suggest that each load should be kept separately. When the cars were to be parked by the houses, they could be driven down the road. But to park them the child would have to move them up on the "grass." The animal group cages would preserve the groups so they could be counted, but the animals were so chosen that no system of classification would serve to help in the solution. In addition, the design of the cages was such that cage building would be highly attractive to most children.

Cargo Groups (see Appendix A)

Measurement division. Ten children at each age level from three to eight years did this problem. From the information, it was possible to determine the kind of general procedure each child used in response to the problem. The observed procedures were arranged in seven categories as follows:

I. The child listens to the problem, looks at the apparatus, and gives the correct solution without manipulation.

II. The child makes groups of three cars, counts the groups, and gives a correct solution.

III. The child places cars on the ferry and makes one or more trips but gives the correct solution before all the cars are moved across the river.

IV. The child places cars on the ferry three at a time, makes five crossings, and gives the correct solution.

V. The child places cars on the ferry, three at a time, makes five crossings, but gives an incorrect solution.

VI. The child places cars on the ferry but not three at a time, gets all cars across the river, and gives an incorrect solution.

VII. The child either does not attempt the task or abandons it before the cars are all across the river.

Of course, these categories might have been collasped into two classes--one for correct solutions and the other for incorrect solutions. Table 1 shows the distribution of responses in the seven categories according to age and sex of the child.

The response categories were listed in order from what was judged to be the highest level of solution observed to the lowest. The distribution of subjects among the response categories indicates a relation between the level of the solution and the age. The striking thing, however, was the great variety of procedures observed to be used in a single age group. The six-year-olds, for example, ranged from being highly manipulative on the task to very low. One six-year-old appeared to have only minimal comprehension of the problem while another displayed almost complete control. The others were spread between these two levels. While the eight-year-olds all gave the correct response, half of them got involved in manipulating the cars across the river to arrive at the solution.

Table 1

Responses at Various Ages

Age in Years	Incorrect Solution (N = 35)				Correct Solution (N = 25)		
8				M F M F	M	F F	F F F
7			M M		M	M M M F	M M M
6		M	M MM M	F M F		F	M
5			M FF M FF M FF	F			
4		M MM M	M F M F M F				
3	M M M	M F F F	M F	F			
	VII	VI	V	IV	III	II	I

Response Categories

Note. F = Females and M = Males.

The most common response categories were those labeled IV and V. In each of these categories the children moved the cars across three at a time. However, some of them could answer how many trips they took and some could not. Those who could not were put in category V. There were almost three times more responses in this category than were in category

IV. Children in both categories used a perfectly valid way of determining the result, but by far the majority of them failed to remember the number of trips they had taken. The problem was structured so that there would not be any record of the number of trips unless the child made a deliberate attempt to group cars as the ferry was unloaded, to keep some mental count, or to use some other means of keeping track of the number of trips. The method of actually moving the cars across the river three at a time peaked at age five and was less for children either older or younger. The three- and four-year-olds apparently had trouble remembering the rules. There was some indication that boys tended to bring their own reality into the situation more often than girls. However, this type of behavior did not appear to be related to the child's ability to give a correct response to the problem.

Although analysis of verbal responses was not central to this study, it would appear that there is something to be gained from such an analysis. About half the participants made no verbal response at all except when they gave a solution. One response which persisted across the age range had to do with a clarification of the rules. There were questions such as, "Do I do it?" "How many cars on the ferry?" "Three and one more?" and the like. The younger children often wanted to make their own rules. For example, there was room on the ferry for four cars, and many of them wanted to put four on instead of three. They would also make rules about what cars should be parked together and how the cars should be loaded and unloaded. Another kind of rule making was in the form, "The yellow car wants to go back." and "This car has to back up."

It was assumed that some of the boys, at least, would make the sounds that go with the movement of cars and ferries. Six boys in the sample did, indeed, make such sounds; they were in the three-, four-, and five-year age range. It was also assumed that counting would be a very common verbal behavior across the age range. There may have been some covert counting, and no doubt there was, but only four children counted so it could be observed and three of them were eight-year-olds. The other was a six-year-old. Some of them may have had trouble making groups of three cars because they could not recognize such a group. Children who made errors on the first load were corrected, but many of them continued to make errors in the number of cars they put on the ferry.

The interesting thing is that the procedure of moving the cars across the river three at a time in a more or less systematic way was observed in children as young as age three but still persisted in the behavior of eight-year-olds.

The procedure described in category III was used by only two children. These children started to move the cars across but apparently realized that they could use a more efficient procedure; they made groups of three cars and gave the number of trips it would take without further manipulation. It would appear that most children, once having undertaken the moving across, did not want to abandon the procedure until it was completed.

At every age level at least one child responded to the reality of the situation in a greater measure than the problem seemed to demand. This behavior was manifested in the child making chugging noises for the ferry, roaring noises to accompany the cars' movement, driving the cars onto and off the ferry, turning the ferry around for docking, and the like. Such behavior was most common among the three-, four-, and five-year-olds. The most common verbal behavior among the younger children appeared to be description of the actions being carried out. As they loaded the ferry and moved it back and forth across the river, they would provide a verbal monitoring of what they were doing. The purpose did not seem to be to communicate with the experimenter but appeared to be merely a verbal equivalent of the actions as they occurred. The behavior persisted through age six but was absent in the seven- and eight- year-olds. One seven-year-old, however, summed up as follows, "Well, you see what you do. You take three, and you get another three and another three until you get to the end and that's how you find how many you got." This child did not move the cars across on the ferry; he just gave the answer and a verbal explanation of his procedure.

Three three-year-olds made comments about the apparatus. They seemed to be more interested in the features of the cars and layout than in the problem. Two three-year-olds and a four-year-old wanted to talk about something entirely unrelated to the situation. A three-year-old girl had trouble with her knee socks which kept falling down and a boy was attracted by a name tag that had been put on his shirt when he entered the room. The four-year-old had just had a birthday and wanted to talk about that. In the cargo group measurement problem, the most important observation seems to be that the process of grouping by threes is available to at least some children in every age range. Over half of the children in the sample actively loaded the ferry and took the cars across. Of those who used this procedure, the great majority failed to remember what the problem was about and could not give a correct solution. In designing instruction, it would seem to be important at some stage at least to arrange for problem situations which would preserve the integrity of groups once a grouping is made. (See discussion under animal groups.)

Partitive division. As soon as the children completed the measurement division task, the partitive division problem was given. They were asked to park all the cars they had just brought across the river around the three houses so that each house would have the same number of cars. The observed procedures were arranged into six categories:

I. The child places five cars at a time at each house or makes groups of five cars.

II. The child distributes cars systematically one at a time among the houses until there are five at each house.

III. The child uses a partly systematic procedure to distribute the cars and arrives at a correct solution.

IV. The child uses a partly systematic procedure to distribute the cars but does not arrive at a correct solution.

V. The child uses an apparent random system of distribution and does not achieve a correct solution.

VI. The child either does not attempt the task or abandons it part way through.

These categories are arranged in an order from what is judged to be the highest to the lowest level of solution. The six categories might be collapsed into two--one for correct solutions (I, II, III) and the other for incorrect solutions (IV, V, VI). Table 2 is a distribution of responses in the six categories according to age and sex of the child.

Although the partitioning process is often considered to be complicated and to require more systematic treatment than measuring out equal groups, there was more success in this problem than in the previous one. Eleven children who could not do the measuring problem could do this one. Only two who were successful in the measurement problem could not do this one. One reason may be that the children were more familiar with the apparatus. A more reasonable explanation seems to be that the child had a record of the groups of cars in front of him once he had moved the cars to the houses. Adjustments could then be made and the feedback used to make the result fit the question that was asked.

The very systematic approach to the partitioning process first showed up in the four-year age group and was used increasingly among older children. Eight-year-olds may have been able to deduce that if it took five trips to get the cars across the river in groups of three, then there would be five cars at each of the three houses. Still, seven of the ten eight-year-olds actually distributed the cars around the houses in a systematic manner, as in dealing cards, and gave the answer only after the distribution had been made. Regardless of the skill in partitioning, there were few children who seemed to see a relation between a mathematical expression of division and the process they were carrying out.

Table 2

Responses at Various Ages

Age in Years	No Solution (N = 26)			Solution (N = 34)		
8					F F F M F M F	F M F
7				M M	M M M M M	M M F
6		M M M	F		M M M F M F	
5			F F F F	M M F	F M F	
4	M M	M M M M M	F	F	F	
3	M F M F M F	M M F	F			
	VI	V	IV	III	II	I

Note. F = Females and M = Males.

Over half the children in the sample made no verbal response while they were solving the problem and only responded to give their solution or when questioned about it. The younger children, aged three to six, tended to ask questions to clarify the problem more than they did in the measurement problem. One of the difficulties with these younger people was that once they had parked one car by each house, they could not see why more cars should be parked there. Closely associated with this concern were questions about how the cars could be parked at the houses. Some children at every age level tried to match the color of a car with the color of the house. For the younger children, this was distracting enough to prevent some of them from solving the problem. Although older children were often distracted by color matching, they could at the same time give their attention to the number of cars at the houses and come to a correct solution. Again, there was very little evidence of overt counting and if it did occur, it was in older children.

In summary, it might be observed that though the partitioning process may in most circumstances have more inherent difficulties than the measuring process in division, there were more solutions for this problem than for the measurement one. The very young children had trouble making sense out of the situation, probably because of the large number of cars that were to be parked and the small number of houses and also because the color of some cars matched the color of some houses. Once these distractions could be overcome this seemed to be the easier problem to solve.

Animal Groups (see Appendix A)

Measurement division. After they had completed the cargo groups, five of the ten children at each age level were given the animal groups problem (see Appendix A for details). In the measurement situation, they were given animals and asked how many cages would be needed if there were five animals in each cage. Although this problem when completed would preserve the integrity of the groups and permit the child to check his response, the choice of animals was such that it would be necessary to put animals together in a cage which would never be together in reality. A further distraction involved the building of the cages which those children seeking a physical solution would have to do.

The response categories for this problem situation were as follows:

I. The child gives a correct solution by making groups of five or by making no physical contact with the material.

II. The child makes one or more cages, puts five animals in each, and continues building cages until there are enough and all animals are in cages, then gives the correct solution.

III. The child makes two or more cages, stops building cages and distributes animals among those cages and gets an incorrect solution.

IV. The task is abandoned or not attempted.

Table 3

Responses at Various Ages

Ages in Years	Incorrect Solution (N = 13)		Correct Solution (N = 17)	
8			M F M F	F
7			M M M F	M
6		M M M F	F	
5		M F	F F F	
4		M M	F M F	
3	M M F	M F		
	IV	III	II	I

Response Categories

Note. F = Females and M = Males.

Only five children out of twenty-seven who attempted the problem appeared to be able to ignore the reality in this situation and to arrive at a solution. Three three-year-olds did not attempt the task. By far the majority of the children were either completely engaged in classifying the animals, making them stand upright, finding ways of getting them into cages, in building cages, or in some combination of these behaviors. Except among the three-year-olds, there were children in every age range who could overcome such distractions and solve the problem.

One question which might be asked is this: Were the children who were successful on the measurement part of the cargo groups problem also successful on this one? It was found that seven children who failed to get the measurement part of the cargo groups problem were successful in doing the measurement problem with the animals. Only one child who was able to do the cargo groups problem failed to get the one with the animal groups.

Only five children in the entire sample were able to ignore the reality aspects of the problem in seeking a solution and three of these were eight-year-olds. All of the others felt compelled to put animals in a cage only if they were of the same kind, or to make animals stand upright, or to remove a panel in the cage for the animal to enter, and the like. This inability to ignore how things should be in reality appeared to be an important source of difficulty at every age level except seven and eight. Some children would have all the animals in the cages, for example, except the lion and would hold it trying to decide what to do with it. Others would get so involved in getting the animals to stand that they seemed to forget what the problem was. Undoubtedly some younger ones did not know how to make a group of five. There was a high incidence of questions about the problem or comments about the apparatus. Children were apparently trying to get information about how to handle the incongruities.

In this problem, once the animals were placed in the cages, all they had to do was count the cages. In the cargo groups problem, they finished the process and did not have anything to count. This fundamental difference in the nature of the two problems appeared to make this one easier to do. It would appear that a very careful analysis of problems is necessary if they are to become an integral part of an instructional program.

Partitive division. As soon as the measurement part of the problem was completed, the child was provided with three cages, eighteen animals, and asked how many animals would be in each cage if each would hold the same number.

The response categories are as follows:

I. The child places six animals in each cage or partitions animals among the cages and gives the correct solution.

II. The child places two or more animals in each cage to begin with, then partitions the remainder one by one and gives a solution.

III. The child places animals in cages in unequal groups, then evens them out and gives a correct solution.

IV. The child places animals in cages, tries to even them out but can't give a correct solution.

V. Task abandoned or not attempted.

Table 4 shows how the responses were distributed.

There were two children who could not do the measurement part of this problem who were successful in this part. Only one child did the measurement but not this part. All the others missed them both or got them both. All of the seven- and eight-year-olds could do both parts. There was a decrease in the frequency of questions or comments about the problem or about the apparatus and an increase in observable counting behavior.

It should be noted that this was the last of four quite distinct problems concerning division for the thirty children. This problem supplied a good deal of feedback in that the cages kept the groups distinct and the child could keep a check on what was happening. There were a number of children who responded to the reality of the situation more than they needed to in order to solve the problem. Indeed, four out of the five eight-year-olds put the animals in cages according to class, made sure animals stood up, arranged the animals in groups, adjusted fallen animals, and the like. This type of behavior was observed to be prevalent at all levels except at age three. The three-year-olds either found the problem incomprehensible or did not attempt it.

This behavior contrasted sharply with that observed for the partitive problem with the cargo groups. For that problem, only one of the eight-year-olds out of the ten showed evidence of responding strongly to the reality of the situation. The others of this level and age, seven in all, just distributed the cars among the houses without apparent regard for type or arrangement or any other physical aspect. Attempts to drive the cars to houses, to group cars of the same color, to match car and house colors, etc., did occur, however, among the three-, four-, five-, and six-year-olds.

Table 4

Ages in Years	No Solution (N = 12)		Correct Solution (N = 18)		
	V	IV	III	II	I
8			M F	F	M F
7				M M M F	M
6		M M F		M F	
5		F F	F	M	F
4		M M	M F	F	
3	M MM F	F			

Response Categories

Note. F = Females and M = Males.

Summary

The following general findings are significant:

1. The problems which were designed to study behaviors associated with measurement and partitive division contained a variety of distractors to which children at all age levels in the sample responded. Children in the seven- and eight-year-old sample were, in general, able to cope with the distractions and to solve the problems. However, the younger children were often so distracted by the reality in the situation that their ability to handle the problem was impaired. A much closer monitoring of the reality aspects of problems will have to be achieved before a proper assessment of the instructional value of such problems can be made. It may be that problems have to be designed which provide a range of distracting elements. In any case, mastery of the process would apparently involve the child in coping with varying amounts of "noise" in the form of such distractions.

2. The partitioning process appears, on analysis, to be more complicated than the measuring process in division. However, the question asked in the measurement situation seems to be a very difficult one if the integrity of the equal groups is not maintained in the process of solution. Thus, children who moved the cars across on the ferry failed often to be able to remember how many trips they had taken--if indeed they gave any thought to the question at all. On the other hand, children had no difficulty with the animals in the cages because the grouping of five animals was maintained throughout and provided a continual check. The process requires that to answer the question the child must remember the number of groups. Here again the design of problems has to be brought into question. Any attempt to establish instructional procedures based on the problems would have to take into account the kind and amount of information, in the form of feedback, provided to the child. It would appear to be important to maintain control of the feedback dimension and to consider its influence on the ability of the child to handle problems in a particular domain. Some very limited examination of the other problems in the investigation would seem to lend some support to this view. In the reflected shapes, for example, the shadows can be made to change constantly and thus provide the child with immediate and constant information about his success. Children obviously use this information to guide them to solutions to problems in a very unfamiliar setting.

3. It was assumed that the seven- and eight-year-olds, at least, would respond to the simple division problems as set up in the protocols with very minimal reference to or contact with the objects and materials in the problems. Rather surprisingly, there was a great deal of dependence on manipulation of the objects among children of all ages in the sample. Even though they might have been able

to respond by just looking at the apparatus, they seemed to want to verify by making the physical moves implied in the statement of the problem by the interviewer. The materials themselves may have been attractive enought to magnify this tendency.

4. Protocols which the interviewer was required to follow when interacting with the child and in setting up the problem made it impossible to check the influence this factor had on the behavior of the child. It would seem that directions such as, "Build enough cages so that there will be five animals in each and so that all the animals are in cages.", would stimulate a different set of responses than, "How many cages would be needed if there were five of these animals in a cage?". In any case, the presentation of the problem to the child would appear to be an important determinant of subsequent problem solving behavior. If it were possible to monitor such presentations, it may be possible to show that the use of symbolic solutions as opposed to purely physical ones could be controlled largely by how the problem was stated.

5. In each situation, the cargo and animal groups, the partitive problem always came after the measurement one. Children more often asked questions about the partitive problem than about the measurement problem. This may have been because the partitive situation, if done systematically, would require more moves and more control than the measurement one. It seems as likely, however, that the requirements of the first problem interfered with the child's understanding of the requirements of the second. In many cases the child would ask what aspects of the new problem were the same as the former one. The confusion might have come because of two apparently different but obviously related problems using the same apparatus. The type and variety of problems involving identical or nearly identical apparatus and their interactions might have to be considered in instructional programs which contain a large proportion of problem solving activities.

The criteria on which the development of these problems and problem situations were based included reference to "multiple embodiment" in the sense that the same mathematical idea should appear in more than one embodiment (Dienes, 1960). The question which has not been considered in the criteria is concerned with the use of a single problem situation to involve a variety of mathematical ideas. Judging from the questions the children asked, use of a single situation in this way may be an additional source of noise and, therefore, an important factor in problem solving behavior.

Discussion

From the foregoing summary it is clear that a number of refinements must be made on the model used to develop problems and the model used to study the problem solving behavior of children from age three to age eight.

The analysis of the data from only one set of related problems has been considered at any length. As analyses for the other problems are completed, it seems reasonable to assume that other guidelines will develop for the inclusion of problem solving in a model for mathematics learning in the range of ages from three to eight. Before this is accomplished, however, work will have to precede this that will eliminate some of the crudities of the present investigation and lead to refinements in the model used for studying problem solving behavior.

The most crucial factors to incorporate in the criteria for good problems or in some other part of the model appear to be the following:

1. A control over distracting aspects of the problem. For learning to take place the child would have to "cut through" a variety of distracting elements. Indeed, mastery of the problem and the processes involved would probably demand it. Thus, one would want to develop, for example, a series of problems in division in which the child could be provided with distractions of various kinds and amounts and to make a systematic study of these distractions and their influence on problem solving behavior.

2. A method of monitoring feedback. Some problems supply intrinsically, continual and immediate feedback. In others, if there is any feedback at all, the child may have to go through a series of involved processes and may actually lose contact with the problem he started to solve. Some method has to be devised to vary the feedback dimension and to maintain a measure of control over it.

3. Form of presenting problems to children. The nature of the apparatus and the verbal communication with the child needs to be carefully monitored to give some assurance that a child, in making a response, is actually responding to the problem the investigator has in mind. There should be an effort made to determine which problem type will stimulate the child to make a more symbolic form of response and which form stimulates a more physical one. Some young children, for example, in the ferrying, were more involved in getting cars on and off the ferry and in guiding the ferry than in determining the number of trips; they did not even consider that problem. Others would probably have said it will take five trips but were interested primarily in the ferrying process so deferred their answer until the action had been completed.

4. Use of a particular apparatus for a variety of problems. The possible difficulty here can perhaps be expressed in many ways. In setting up the problems, it was assumed that a particular piece of apparatus was as effective a vehicle for one kind of problem as another. The evidence, meager as it is at this point, is that such may not be the case. There appears to be some possibility, whether advantageous or not, that using an apparatus for one problem may cause some uncontrolled interactions. This needs to be more carefully accounted for.

Up to this point, only limited analysis has been completed. Indications are that the model for studying problem solving behaviors is sufficiently productive to warrant further refinement. Fortunately, the information seems to point the way to what general areas need to be improved. Any research on problem solving, whether in this model or not, should take into account the general areas which have been discussed here.

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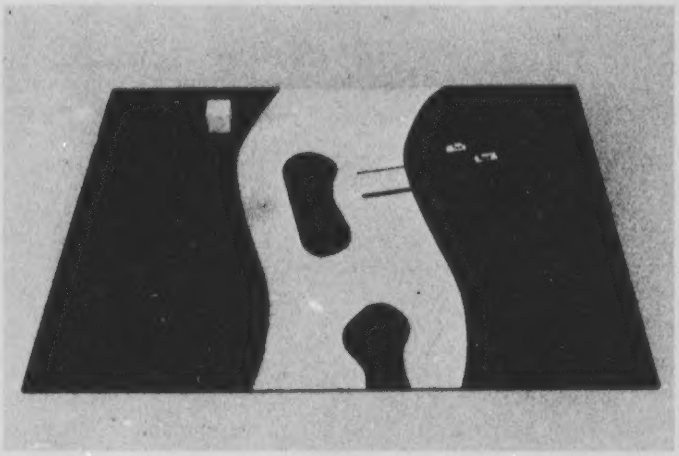
Appendix A

Measurement and Partitive Division

Cargo Groups

The apparatus is placed on a table in front of the child who is standing or seated. The model is referred to and the child is shown the river, the islands, the ferry boat, the parking lots, and the houses. There are fifteen plastic cars in the parking lot on one side of the river and three houses placed on the other side of the river. The child is first shown how the ferry can cross the river and is asked to choose a car, put it on the ferry, take it off the ferry, and park it in the parking lot on that side. Assistance is given the child with these moves if necessary. When they are completed, the car is returned to the first parking lot.

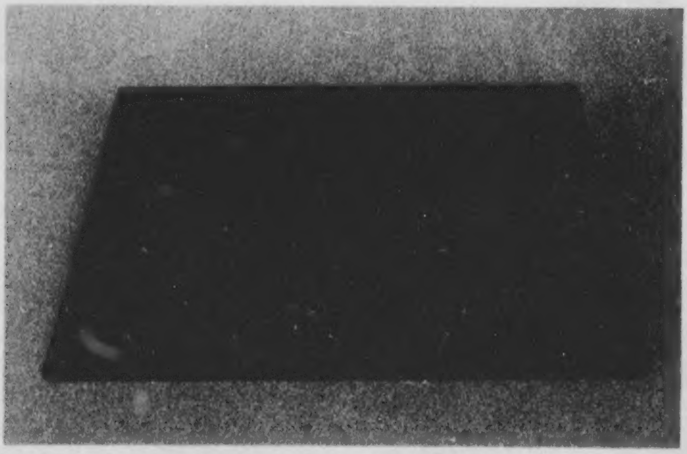


The child is then told that all the cars are to be taken across the river and parked in the second parking lot. He is advised that the ferry accommodates exactly three cars each trip and the following question is asked: "If the ferry boat can take only three cars each trip, how many trips must the ferry take to get all the cars across?" If the child does not appear to be interacting at any level, then after a period of about fifteen seconds ask: "Would you like me to repeat the question?" and repeat as necesary. If the child still appears to be stymied and unable to give a response, suggest that he can load the cars on the ferry and take them across if he wants to. If the child loads other than three cars on the ferry ask: "How many cars were you supposed to put on the ferry?" and give the answer as necessary. Provide this information only once. If he persists in putting other than three cars on the ferry, further help is restricted to a reminder of the original question. When the child indicates that he has finished the operation he is asked: "How many trips did the ferry boat take?" If all the cars are not on the second parking lot, they are now assembled there.

The child is asked then to park all the cars beside the three houses so that there are the same number of cars at each house. If the partitioning operation offers some difficulty, he is reminded of the original problem. When the child has parked all the cars, he is asked: "Does each house have the same number of cars? "How many cars at each house?" Be sure to ask these questions in the same order for all children.

Animal Groups

The board is placed on a low table in front of the child so that he has an overview of it. Two boxes are placed between the child and the board: one containing 26 posts and 17 slats for fence building and another containing an assortment of 20 toy animals. The child is shown how the posts fit into the holes, which were designed to accommodate them, on the board. He is also shown how a slat fits between two posts to make a fence. The child is asked to build two more fences, like the one he was shown, anywhere on the board. When the child has completed these, he is asked to build a closed corral (cage, pen) to keep some animals in. If he has any difficulty constructing this cage, he is given assistance.



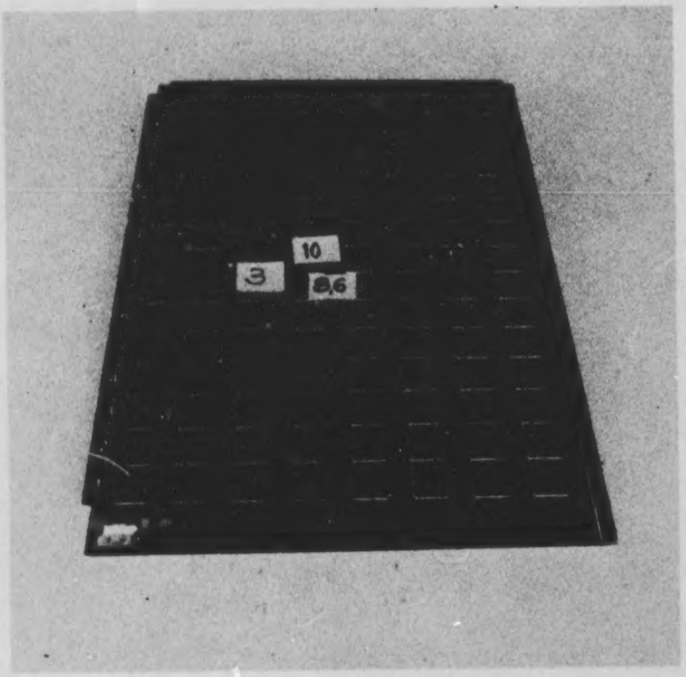
The child is then presented with the box of animals which includes four camels, four ducks, four mice, four hippopotamuses, an elephant, moose, horse, and a lion. The child is asked: "If each cage holds five animals how many cages will we need?" If the child does not appear to be interacting at any level after an interval of approximately fifteen seconds ask, "Would you like me to repeat the question?" and repeat as necessary. If the child still appears to be stymied and unable to make a response suggest that he build the cages and put the animals in them. If the child puts other than five animals in each cage ask, "How many animals were supposed to go in each cage?" and give the answer as necessary. If he still persists in putting other than five animals per cage he is shown a set of five. Further help is limited to a reminder of the original question. When he indicates that he has finished the task, the child is asked, "Are there the same number in each cage?" "How many cages are there?"

The animals are collected, two camels are removed, and the remainder (18) are placed in the box. One of the cages is dismantled so only three cages remain. The child is told that the remaining cages are for the animals in the box. Ask the symbolic questions first. He is asked to put all the animals in the cages so that there are the same number of animals in each cage. If the child has difficulty with this operation or appears to have forgotten the problem, he is reminded of the original question. When the child appears to have completed the task to his satisfaction he is asked, "Are there the same number in each cage?" "How many animals are there in each cage?" The order in asking these questions should be the same for all children.

Appendix B

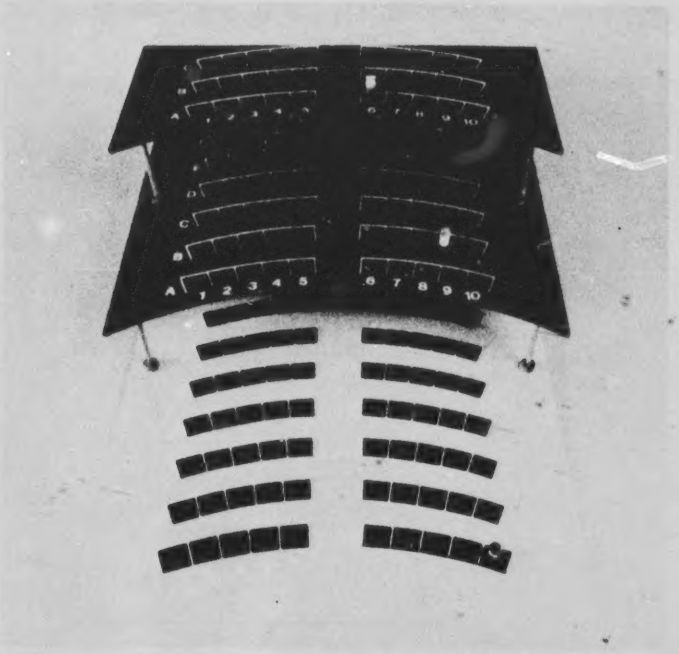
Coordinated Reference Systems

Parking Lot Grid



On the simulated parking lot, the child is required to park cars as indicated on a slip of paper. There is an ordered pair shown on the slip. The first number indicates how many spaces to the right; the second how many up. A warm-up exercise is provided. The object is to determine how a child coordinates moves in two dimensions.

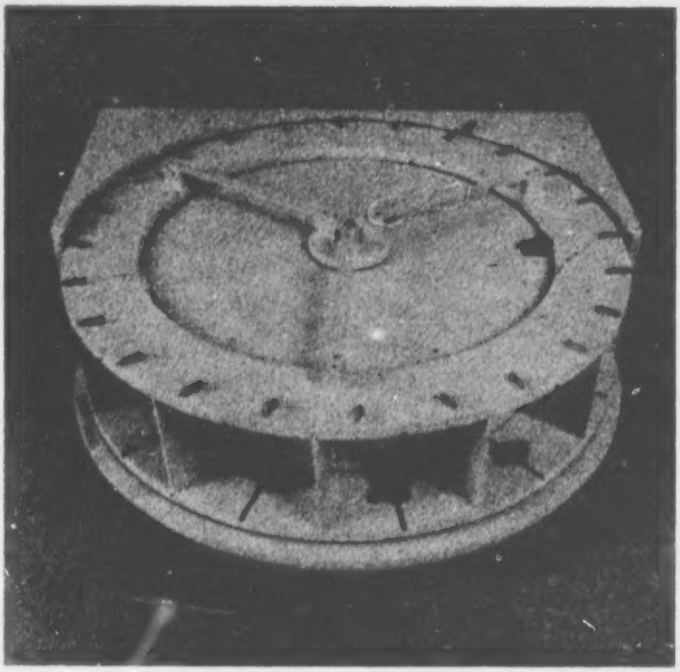
Theatre Grid



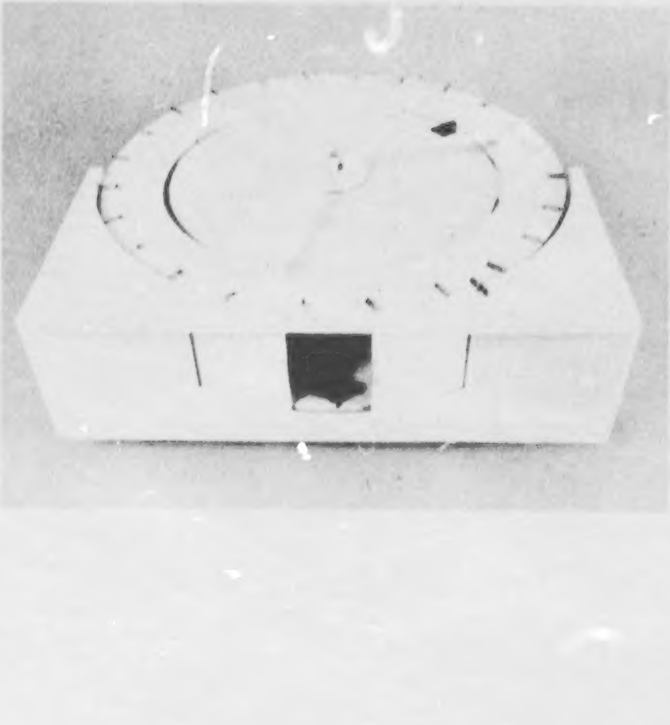
A theatre with three stages is presented to the child. Each stage is a different color. Seats on each stage are labeled with letters for rows and numerals for seats in the row. The child is given a ticket with a letter and a numeral written on it. The color of the ink used to write the letter and the numeral indicates the floor. He is then given a wooden man and asked to place the man in the correct seat. The object is to determine how the child coordinates moves in three dimensions.

Sequences

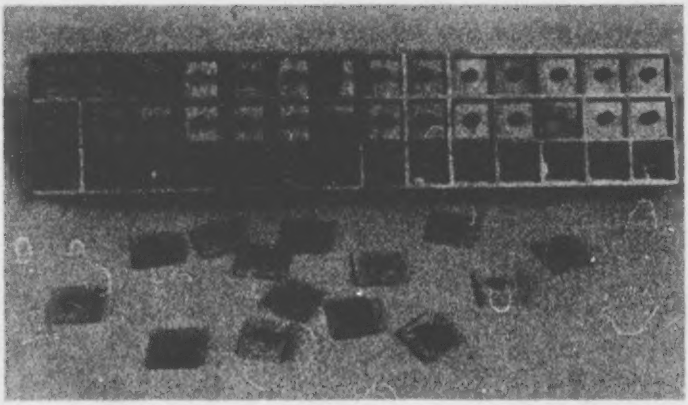
Circular Sequence



The carousel in which sequences are displayed has twelve divisions. The back is open so that objects placed in divisions can be viewed. The front is arranged so that the child can only see one division at a time. He is shown objects in several of the divisions from the front in order as the carousel is turned. The sequence set in the carousel is as follows: blue airplane, camel, red airplane, camel, red airplane, camel, blue airplane, elephant, red airplane, elephant, red airplane, elephant. After seeing a few of the objects he is required to predict what will come next. The nature of these predictions is assumed to give some indication of his knowledge of the sequence. He is permitted to go around back and see how the objects are placed in the carousel and then is given some objects and asked to make his own sequence.



Linear Sequence

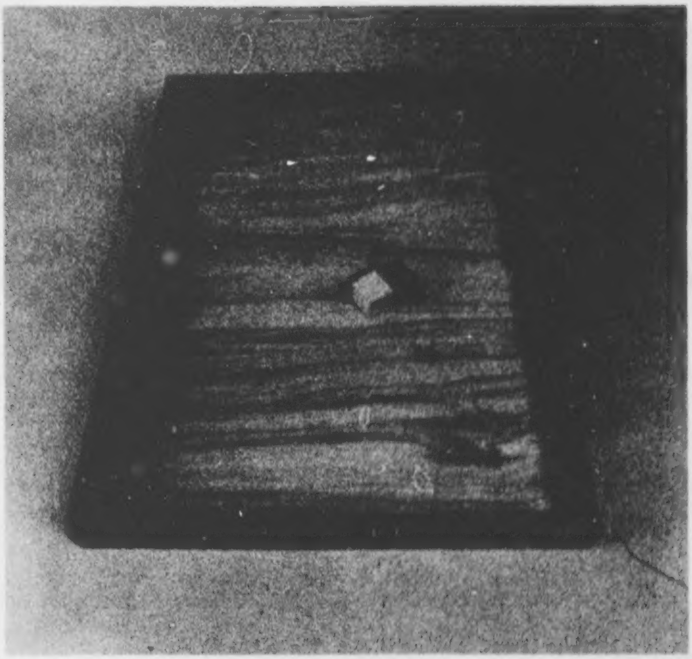


A box with three rows of 14 compartments is presented. Each compartment has a snug-fitting cover. The covers on the row nearest the child are all off. Those on the other two rows are left on. In the middle row the compartments each have a small colored block in them in this order: orange, brown, orange, brown, orange, brown, etc. The back row has colored blocks but in this order: orange, orange, brown, blue, orange, orange, brown, blue, etc.

The child is shown what is in the first three compartments of each row in turn and then asked to predict what is in the successive compartments. A prediction must be made before the cover is removed. The final problem is for the child to place objects in the row closest to him in some order. The covers are placed on the compartments and the child asked to predict what is in each.

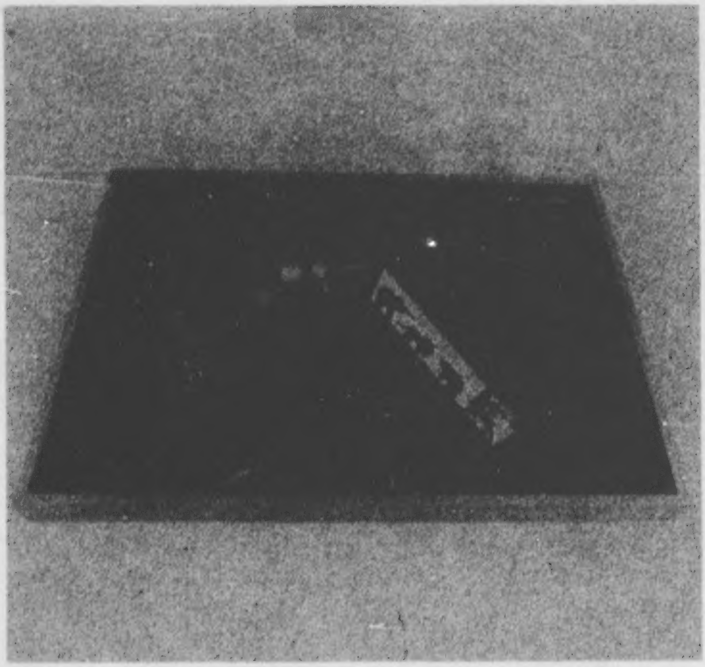
Reflections on a Plane

Object Reflection



The board shown has rubber bumpers along its edges. A shooter ejects a steel ball which can be bounced off the bumpers. An object (in this case, a plastic bear) is placed on the board and the child asked to knock it over by shooting the steel ball. Then some blocks representing houses are placed between the object and the shooter. The child is instructed to bounce the ball off one side and knock over the object. Finally the object is placed behind the houses so that two bounces are necessary. The way the child aims and makes corrections in pointing the shooter is the behavior most closely monitored.

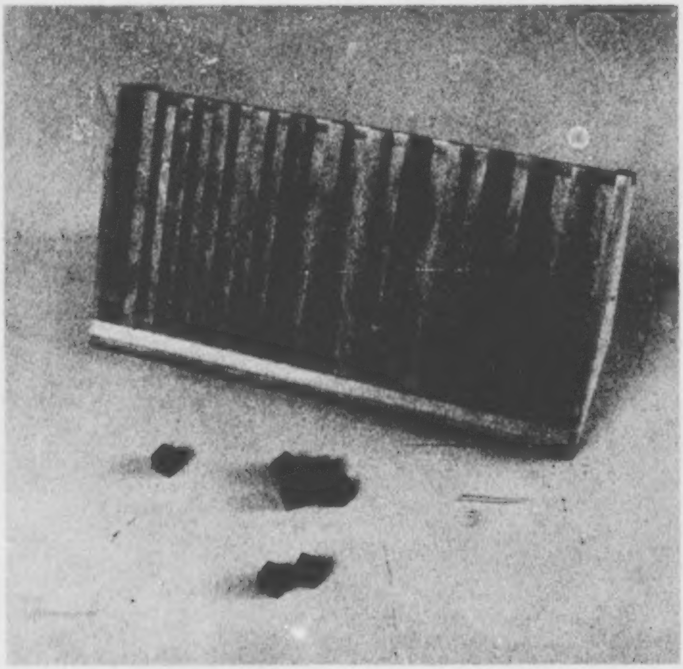
Mirror Reflection



This board is similar to the one for object reflections except the edges have mirrors all around. The shooter, in this case, is a point source of light. The beam can be trained on silhouettes of animals. Here again a direct hit, one reflection, and two reflections are required in the problem. The way the child controls the light source is of particular interest.

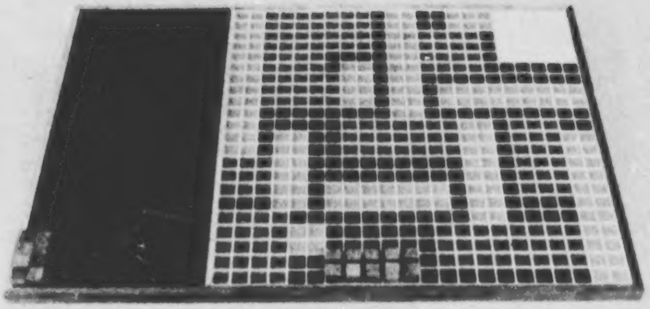
Factors

Factor Platform



The board has thirteen vertical grooves into which small blocks will fit. Twelve blocks are arranged in four grooves: three in the first, one in the second, six in the third, and two in the fourth. The child is asked whether the blocks can be arranged so there will be the same number of blocks in each pile. The number of blocks and the number of rows occupied by blocks are varied. The main object is to determine whether children see any connection between the number of blocks and the way they can be arranged in the grooves.

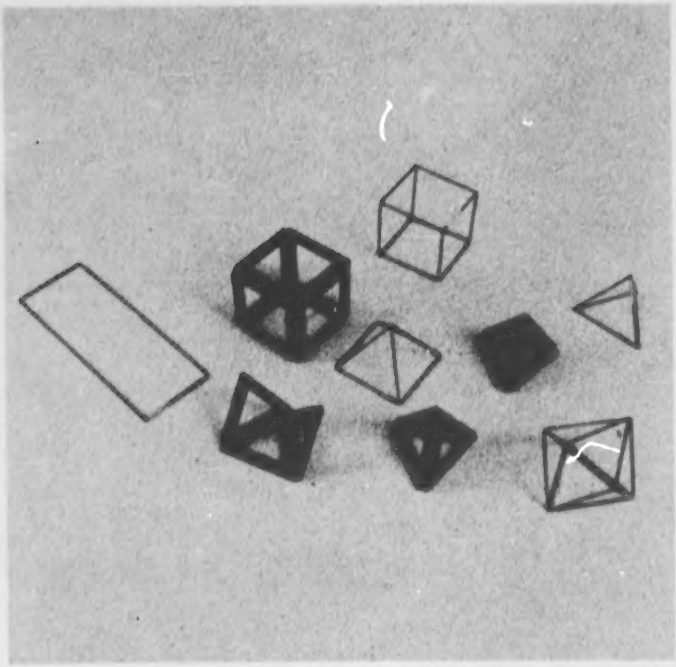
Factor Board



Rectangular areas are indicated on the board in different colors. Rectangles made up of a number of squares divisible by two are in yellow, divisible by three in blue, divisible by four in red. The child is given a number of blocks and asked to find a colored space that they will just fit. He is given two more blocks and asked to find another which the whole set will fit. The object is to find whether the child sees any relation between the number of blocks he has and the factors of that number. It is expected that if he could see such a relationship his search strategy should reflect it.

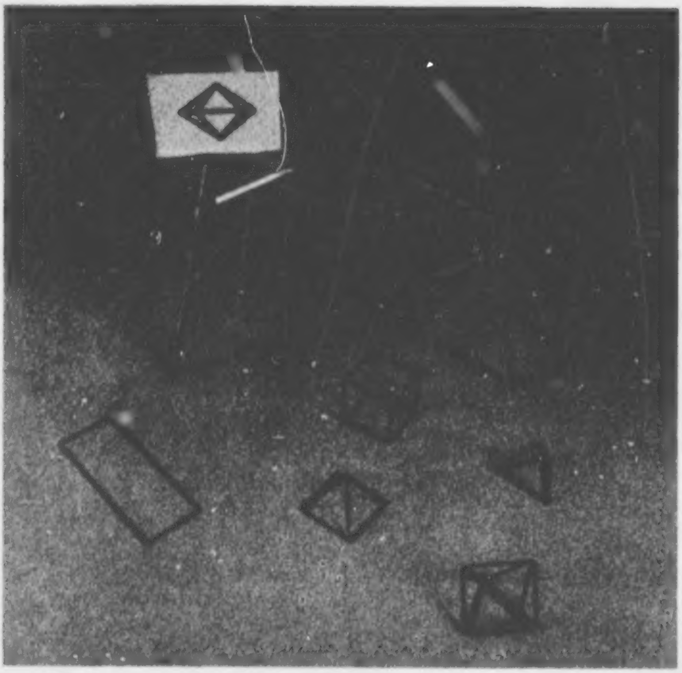
Geometrical Representation

Fold-out Shapes



Squares, triangles, and pentagons are fitted with alternating strips of Velcro so that various geometric shapes can be easily constructed and dismantled. The child is shown a cube and how it can be dismantled. He is asked to reconstruct the dismantled cube. In the main problem he is shown the cube dismantled with the pieces joined in various ways and asked whether it can be folded to make a "box" without changing the relationship of any of the pieces. Similar tasks are required for the tetrahedron. Finally, the child is shown a dodecahedron and is asked first to lay it out flat and then to reconstruct it. Particular note is made of the predictions the child makes and the kind of moves he uses to reconstruct a shape.

Projected Shapes



A set of wire networks is available for the child. A screen is provided so that the shadows of the shapes can be projected on it. In one corner of the screen a diagram of a shadow is shown. The child has to choose the particular wire shape and turn it so as to make a copy of the diagram. The choices made by the child and the moves he makes to copy the diagram with the shadow are of special interest in this problem.

Linguistic, Logical and Cognitive Models for Learning Mathematical Concepts[1]

Harry Beilin

City University of New York/Graduate School

The conduct of research in mathematics learning is dependent both upon the nature of mathematics and the nature of the learner.[2] Considering that mathematics is a complex discipline that encompasses many subject matters, its properties are not likely to be subsumed by a single conceptual category or by one conceptual system. It follows that no single model for research is likely to be adequate, and the purpose of this paper is to show that a variety of models are required to reflect adequately the varied functions of mathematics. Knowing the nature of mathematics is not in itself sufficient, however, to define the scope and focus of mathematics learning. What is needed additionally is a specification of those features of thinking and learning that are required irrespective of the nature of the subject matter. Some features are constrained by the nature of mathematics, but others are independent of it.

The Nature Of Mathematics

It might be said that mathematics is a theory (or a set of theories) about the nature of reality. The classical empiricists, in particular, have maintained that ideas of reality derive primarily, if not solely, from experience. Such mathematical notions, for example, as "infinite structure" are said by them to be too ambiguous to be true or even useful if they do not in fact make reference to the real world (Benacerraf & Putnam, 1964). If the physical world is the source of such mathematical concepts (as "infinite structure"), one should be able to look to physics for such legitimacy. Hilbert, as is well known, argued that physics could offer no such security since the evidence from physics for an infinite and continuous universe has been progressively eroded by arguments for finiteness and discontinuity. Hilbert held that if mathematics were not to be reduced to reliance on dubious physical assumptions, then its own assumptions had to be independent of knowledge of the physical world. This view led Russell in turn (in the second edition of Principia Mathematica) to argue that mathematics is concerned not with actual physical existence but only with the possibility of physical existence (Benacerraf & Putnam, 1964, Introduction). Thus, it can be said, on the one hand, that mathematics is an abstract conception of the world, and on the other that it is a hypothetical conception of a possible world. In each case, mathematics is a theory about the world, whether empirically based or hypothetical.

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[1] I am indebted to Professor Walter Prenowitz for a number of insights into mathematics that are reflected in various ways in this paper.

[2] Our discussion is not directly concerned with curriculum or instructional processes; these encompass more than the issues which we treat. The limitations imposed here are motivated by theoretical and pragmatic justifications.

The position that has come to be known as logicism (derived from Frege, Russell, and Whitehead), which is consistent with the hypothetical view of mathematical theory, holds too that mathematics does not have a subject matter but deals instead with the "pure relations among concepts" (Benacerraf & Putnam, 1964, p. 9), that is, such concepts are bound by logical and not empirical relations. By contrast, mathematicians such as Hilbert maintain that mathematics does have an "extralogical" subject-matter, which Hilbert called "expressions" employing elements such as "strokes" (/, // ,///) that are finite, discriminable, and self-evident. Hilbert notwithstanding, the logicist achievement profoundly affected the conception of mathematics by not only axiomatizing much of the existing mathematics, but by attempting to reduce all mathematics to logic.[3] As Benacerraf and Putnam point out, it is now generally regarded that the logicists succeeded in reducing mathematics to elementary logic plus the theory of sets (1964, Introduction) even though it is evident from the proofs of Gödel and others since that the completeness of such a system can no longer be assumed. An important exception to the logicist position is the claim of intuitionist mathematicians such as Brouwer that some of the fundamental concepts and operations of mathematics, such as the system of deduction, involve finitely iterable operations that are purely mathematical and do not belong to logic. Consequently, mathematics can be considered either as a theory about the world that is logically formulated or intuitively known from its operations.

In his early work, Wittgenstein, following Russell and Frege, held that mathematics was reducible to logic and that logic in turn reduced to the propositional calculus. The relation of propositions to mathematical "truth," however, is debated by two groups of mathematicians, the Platonists and non-Platonists. Platonists consider that propositions represent the discovery of structures that have an existence independent of the mind of the mathematician. Non-Platonists, in general, hold that mathematics is a constructive activity in which the mathematician actively creates a system or theory in which propositions are "true" only to the extent they follow from assumptions and definitions proposed by the mathematician and gain acceptance from others by convention. In an extreme form of the non-Platonist position, mathematics is a sort of "language-game," in Wittgenstein's sense, wherein the process of deriving a theorem, for example, results in a new rule of language, and thus mathematical concepts embodied in the theorem undergo change as a proof is developed (Dummett, 1964, p. 496).

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[3] I am indebted to Professor Osborne for pointing out that in Hilbert's view problems and questions were the very source of mathematical ideas, and his formalism was, in part, motivated by a desire to resolve many of the developing conflicts between intuitionism and logical positivism.

The emphasis on language in Wittgenstein's usage serves a radical constructionist thesis and informs to some extent an "empiricist philosophy of mathematics" (Dummett, 1964, p. 503). A similarly central place is given to language in mathematical theory in the views of the logical positivists, particularly Carnap. Formal systems such as logic and mathematics are treated by them as conceptual or theoretical languages, particularly when they appear in scientific theory construction. Thus, at least in some sense, mathematics serves a linguistic function. At one extreme, mathematics is treated as a language system itself, at the other, it is a theory represented by a special (i.e., formal) language.

In attempting to understand the nature of mathematics, one cannot ignore either that mathematical theories appear in contexts that are nonmathematical (in such physical theories as the laws of relativity, magnetism, celestial mechanics, etc.). Such applications of mathematics are often as important to an understanding of mathematics as they are to an understanding of physical and related theories. These contexts may be as diverse as arithmetic (in which the fundamental properties of algebra are applied) or physical theories (in which Lobachevskian or Riemannian geometry are applied). In quite another sense, mathematics serves a "computational" function that makes applied areas more precise, abstract, manipulable and amenable to deductive and inductive inference.

In sum, mathematics is a theory either of reality or a possible model of reality, which is either discovered or constructed (in the way number relations could be said to have been "discovered" in ancient civilizations), and which is represented either in special language structures or functions conventionally as a language (i.e., as a cultural convention). Lastly, mathematics can serve a computational and conceptual function in relation to physical and social reality.

We will consider each of these features of mathematics as a model for research in mathematics learning. Detailing these (or similar) models can serve the researcher in the following ways. First, a model organizes in a systematic and (hopefully) coherent fashion a single conceptual framework for explaining the nature of observed behavior or implicit causal mechanisms. Second, a model serves to provide the basis for testing specific hypotheses about mathematics learning. Third, a model provides an accessible means for applying more abstract theories (such as linguistic or cognitive theories) to observed data than is offered by such theories

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themselves.

A Logical Model

From what has been said, mathematics may be conceived of as a theory in two respects: as a logical theory of axiomatized abstract relations that are purely hypothetical and whose validity depends solely on its consistent and noncontradictory derivation from its premises (in a pre-Gödel sense), and as a theory of the world, again, as an abstract axiomatized system that maps onto perceived physical relations and whose validity is dependent, at least in part, upon systematic relations in reality. The two notions parallel to some extent the Kantian distinction between the analytic and the synthetic. A way of seeing the distinction is to consider two views of geometry considered as a model for explaining and comprehending the nature of space. With respect to geometry as a deductive system (in the logicist sense), one asks whether the theorems of geometry follow logically from its axioms. This is ordinarily the mathematician's task and requires knowledge of relevant logical principles embodied in a mathematical framework. The second sort of question is addressed to whether the axioms of geometry and the theorems derived from them are factually true (i.e., fit reality). Discovering the answer to this question is usually the task of the physicist. In this context, mathematics is applied or used in a scientific theory and is not a "mathematical theory" as such at all (Nagel, 1961).

In the first case, the validity of the logical derivation of the theorems of geometry from its axioms does not depend on the particular meaning of the terms appearing in the premises and the conclusions. The validity of the derivations is dependent instead on the formal structure of the statements that include the terms and on the appropriate use of the logic. Mathematical expressions of this kind entail the use of words or symbols that represent logical relations or operations (Nagel, 1961). The central question that one asks in this context is whether the conclusions (theorems, etc.) follow consistently from the premises (axioms, etc.) and less so as to whether they are true or false.[4] Although the example is from geometry, it applies to all deductive argument in mathematics. Geometry as a logical theory is usually identified as "pure geometry," and when studied as a system of factual validity is known as "applied" or "physical geometry" (Nagel, 1961, p. 221). The attempt to formulate geometry, to stay with this example, as a rigorous logical discipline is evident in Oswald Veblen's 1904 axiomatization of geometry.

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[4] A great deal of attention is given these days (again following Gödel), to questions of decidability: whether and how mathematical relations of particular kinds are capable of proof (i.e., logical deduction).

A contemporary extension of this effort is to be found in Prenowitz and Jordan (1965). In this formulation, the logical elaboration of theorems is based on the theory of sets. The postulates of the theory are based on a series of properties, such as extension, determination, linearity, and dimensionality of a set of primitives identified as a point, line, and plane. The primitives or basic terms are undefined and can remain so because they refer to no physical objects as such (i.e., they are abstract hypothetical entities). The emphasis in the derivation of the theory is on the properties of the logical relations among the primitive terms as explained in the postulates (e.g., a line is a set of points, containing at least two points) and not on any physical or material content. To emphasize that the relations among the primitives are about hypothetical entities and not actual, it is possible and even desirable to employ logical signs or symbols to substitute for words (such as point) since such words have no specific reference.

A logical theory not only employs abstract signifiers (which may be sets or other constructs), but a set of logical operators such as identity, nonidentity, and negation that also represent abstract relations and not actual cognitive operations (i.e., of the mind). The logical operators together with sets thus become the fundamental logical elements in the construction of geometry and mathematics in general, at least as proposed by Russell and the logicists. A theorem (Prenowitz & Jordan, 1965), for example such as, "Two distinct lines have at least one point in common," is thus derived utilizing the postulates of the theory and the operators and terms embodied in them.

The description of the system properties that hold for geometry are applicable to other theories of mathematics, such as algebra. What this description stresses is the fundamental logical character of mathematics not only considered in its "pure" sense but also, at least in part, considered in its applied usuage.

What follows from the foregoing is a conception of mathematics learning based on the logical properties of mathematical theories. However one conceives of the nature of the learning process, it should take account of the fact that mathematics is at root a logical system involving deductive processes. Contemporary mathematics in the broadest sense is not one system; one should speak of algebras, geometries, and the like. Contemporary mathematics embodies a number of theories quite diverse in scope; yet, each entails the use of logical relations and sets. Their diversity results from the elements to which the fundamental postulates refer that may differ from one system of mathematics to another to a very considerable extent. By the same token, the logics that enter into mathematics, although they may not be equally diverse, also differ from one another in some fundamental respects. That is, contemporary logic is constituted by a set of logical theories, each based on different assumptions, utilizing different systems of notation and to some extent differing in application. Thus, a logical model is required that encompasses not only algebraic or

Boolean logic but modal and other logics as well.

It is possible to have a logical model for mathematics learning that emphasizes only the logical properties of mathematics and another that also defines or constrains the properties of mind or thought that are necessary to mathematics learning and problem solving. With the purely logical model, concern is primarily with the logical structure of mathematical tasks (theory, problem, proof, etc.) and alternative ways of structuring mathematical materials for learning. Such a course has been followed in the past and into the present. It defines curriculum development and instructional methods solely on the basis of the logic or mathematical relations inherent in mathematics itself. Many of the so-called "new math" programs were devised on this basis. The primary consideration that determined what was to be included in early mathematics instruction, such as sets, set notation, and set logic was its logical or mathematical relation to the content of more advanced mathematical subject areas This application of the logical model has its virtues because it bears upon critical properties of mathematics itself, and it forces curriculum development to articulate with underlying mathematical systems. Its principal limitation is that it omits consideration of the cognitive characteristics of the learner or else makes implicit assumptions about him that may or may not be correct, and if incorrect may place serious limitation on learning.

The alternative logical model is a conception that translates the delineated model of mathematics into a concern for how learners acquire the ability to deal with theories of a logical nature (in contrast to other disciplines, for example, that bear upon the child's ability to deal with physical facts and generalizations). The latter extended logical model overcomes the limitations of a pure logical model and encompasses both the nature of mathematics and the cognitive properties of the learner. We will see later that this "extended" logical model relates in a significant way to cognitive models of mathematics learning.

In contrast to the emphasis on its logical forms, it should be recognized too that mathematics had its origin in the need for measurement and calculation. Consequently, a complex system of measurement and calculation evolved that is evident in its more sophisticated forms in scaling theory, statistics, and so on. The mathematics associated with these applications is the "pure" mathematics referred to previously, such as the theory of number, probability, etc. There is also a body of mathematical relations and concepts unique to each measurement or calculation system, defined by the physical parameters of that system. Just as natural and formal languages provide contexts in which mathematical ideas and processes are communicated, measurement and calculating systems in which mathematics is applied provide interesting and significant contexts for the study of mathematical concepts.

Thus for practical reasons, in the sense that mathematics has clearly utilitarian value to those with knowledge of it, and for theoretical reasons, in that it may illuminate the nature of mathematical relations and concepts, it is desirable to investigate the nature of mathematics learning employing computational and measurement procedures. The first step is to elucidate the logical and mathematical properties basic to measurement and computation, then to define the specific mathematical contexts in which they appear, and finally to investigate the cognitive capacities that are required to deal with these applied system properties. Thus, again we see that an "extended" logical model is required that takes into account not only the logical form of mathematics and its applications but also the cognitive status of the learner.

Applications of the model. The logical model for learning proposed here has implicit cognitive assumptions, as is evident from the questions the model suggests: How, for example, does information from the world become a concept in respect to the world, if it is correct to view mathematics as a theory of reality? In turn, how do abstract hypothetical concepts come into being either from information from the world or from other concepts? How are systems of such concepts elaborated; what is the relation between individual concepts to logical structures that embody groups of concepts? Or, how does a system become a system, that is, how are the parts built into a structure? What is the relation between logical processes and logical products? That is, what relationship exists between the knowledge one constructs or abstracts to the processes of obtaining such knowledge? Do logical products in the form of logical structures feed back, leading to their use in mathematical reasoning, or are the products of thought independent of the processes that give rise to them? What is the nature of the processes by which logical comparisons of structures occur that lead to isomorphisms, correspondences, etc? The way these questions are formulated suggests a relation between formal systems such as mathematics and cognitive processes and structures. There are, however, questions that concern the formal properties of mathematics as they bear upon mathematics learning.

Can one speak of fundamental mathematical ideas that necessarily precede other mathematical ideas, and would these constrain the learning of mathematics? Is each mathematical discipline in fact logically unique or do they share a common logic? If they share structures or have common properties, what makes them different? Do common properties suggest a common logic? If there are common properties and a common logic, should these be taught prior to differentiated system properties, or would it be more advisable to instruct in each discipline first and then have the learner abstract the common properties? Does the relation between pure and applied mathematics constrain learning in any way? Should applied mathematics be learned prior to pure mathematics or should it be the reverse? How should a mathematical discipline be segmented for most effective learning; are there natural divisions or must they be arbitrary? Is there a minimal unit in mathematics; is it the same in all mathematical disciplines? Is it best to teach in relation to minimal units, or to related units and if the latter, at what level of integration?

Prior to examining the properties of a cognitive model that articulate with the logical model, we will elucidate the properties of a linguistic model for mathematics learning.[5]

A Linguistic Model

The principal assumption of the linguistic model is that the language (or languages) used to represent mathematical theory have properties that determine at least in part the nature of mathematics learning. There are in fact two languages in which mathematics is represented. The first is natural language. One has such natural language expressions as: If two planes have one point in common, they have a second point in common. This sentence has properties common to all natural language sentences in that its grammatical constituents obey linguistic rules that govern the manner in which a sentence is generated. Some rules define how linguistic constituents are combined, while other rules define the roles that particular lexical items (words) can play in a sentence including those that define how words are allowed in specific sentence slots and so on.[6] These linguistic rule systems are quite complex, at least for the linguist to describe, although 2- to 5-year-old children learn their mother language without any special instruction. We indicated in earlier discussion that sentences in geometry contain terms (words) that are not to be understood in the manner that they ordinarily appear in the natural language lexicon. "Plane" is not the usually understood plane and "point" is not the usually understood point. They reflect mathematical and logical properties that are part of a more abstract "meaning" system. By virtue of this, these and related terms require representation in a special language. Such representation is found in the formal or logical language of mathematics that employs a symbol (or more properly a sign) system different from that of natural language. An example of one (algebraic) statement in that language is: $(y_1 - y_2)x + (x_2 - y_1)y + (x_1y_2 - x_2y_1) = 0$.

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[5] The logical model discussed here does not necessarily refer to the process by which mathematical ideas are developed or the way in which mathematics is "done." Even the creative mathematician does not of necessity proceed "logically" in his own thinking. The logical model presented is based on the analysis of mathematical ideas or products of thought. The mathematician like the learner proceeds more "intuitively," following hunches, testing out hypotheses, and devising and revising strategies. These suggest the cognitive processes by which mathematical ideas are constructed. Underlying these activities are systems of thought with abstract formal properties similar in kind if not isomorphic with the logic of the finished mathematical products.

[6] This description reflects the views of the transformational generative linguists, such as Chomsky, 1965. (See Greene (1972) or Dale (1972) for details of this and related linguistic theories.)

This statement could be translated into a natural language statement, but it would be much more awkward to state and even to conceptualize. The virtue of special or formal language is that it more precisely represents mathematical relations and lends itself more easily by the nature of its abstract form to the representation of abstract "ideas." It permits more efficient deduction by the parsimonious expressions of only the terms and operations that enter into the relations, which natural language is unable to do equally well. Are special languages such as mathematics, languages in the same sense as natural languages? The answer is probably yes and no. Each mathematical language can be said to have a lexicon and a syntax. In algebra one would have terms in the lexicon in two classes, variables (x,y,z) and constants (such as pi). In this lexicon there are no "words" with rich denotative and connotative meanings. They are, rather, terms without "meaning" in the same sense as in the logical model. What meaning there is comes from the expression of the relations among these terms, embodied in the various logical operations of the sentence, +, =, -, and the brackets. Again, meaning is different in kind from natural language meaning. What meaning it has comes from "logical meaning," if one can legitimately entertain such a notion. When mathematical expressions contain terms that are "interpreted" empirically, that is, the variables refer to statements about physical reality, then the same terms take on a very different kind of meaning (i.e., referential meaning). In different mathematical theories (geometry in contrast to algebra, for example), the "terms" of the theory likewise differ. Thus, point, line, and plane in geometry, although they can be translated into algebraic terms and are treated with the same logical operations, are represented in a completely different special language, namely the language of ideomorphic or pictorial signs. The "meaning" of such a system by its potential reference to objects or relations in space has "meaning" that differs from the uninterpreted variables and constants of algebra. Thus, x in a statement has no meaning except as defined by the context of the statement. If one substitutes a number or the word "line" for x, it acquires the meaning associated with that number or the meaning of "line" either in its common sense meaning or its special mathematical meaning. Thus mathematics, as a theory of reality, provides a meaningful context for mathematical statements. Mathematical theory as a formalism provides only what could be called formal meaning or the meaning entailed by the assumptions of the formal system. Meaning in these two senses is to be distinguished further from "psychological" meaning--what something "means" to a child or adult, which entails additional properties of meaning. The need for different systems of representation relates at least in part to the referential functions they serve. Number terms differ from spatial terms because they have reference to different aspects of reality.

The fact that mathematics is taught and/or learned in both natural and special language suggests that learning mathematics is in part a linguistic phenomenon. If so, the significance of linguistic representation is that it enables the learner, as well as the mathematician, to employ a means by which mathematical reasoning can occur. The implication of this is that such thought occurs linguistically or by linguistic means. Two possibilities exist. The first is that thought is structured as language is structured and functions similarly. The so-called Whorfian hypothesis is one such view, although Chomsky (1968) has a rather different view of this relation. Second, language (of mathematics) is merely the vehicle for abstract thought. Although such thought is not structured as language is structured, it is facilitated by the forms and functions of language. Piaget holds this view. Viewing the mind as operating by the properties of a language leads to an emphasis on the linguistic features of mathematics learning. It leads further to an examination of the linguistic characteristics of mathematical expressions, how mathematical expressions are constructed and understood, and on the linguistic rule system that governs the generation of such expressions.

On the other hand, the consequence of conceptualizing mathematical languages as representational systems for mathematical thinking is to see them as vehicles for thought and not as the necessary elements in thought. Mathematical language acquisition in this sense facilitates or inhibits mathematical learning; it is not sufficient for ensuring mathematical reasoning. Even though the distinctive properties of linguistic representation may not directly conform to intellectual or cognitive structures, their form may nevertheless affect mathematical understanding. For example, the comprehension of lexical terms, such as number-words, by young children, or the lack of it, may affect their understanding of mathematical relations (Beilin, 1975). Similarly, a lack of understanding of the abstract nature of geometric terms by older children and even adults may adversely affect their understanding of geometry. In addition to the lexicon, the very manner in which mathematical ideas are expressed in natural language statements, which is the mode of representation and communication of much of early mathematics, may differentially affect understanding. We can illustrate this with the following two mathematical expressions that are usually thought to be mathematically equivalent but are probably not equally understood by young children.

If I have ten bananas and take away four bananas, how many do I have left?

If I take away four bananas from ten bananas, how many do I have left?

The second expression is likely to be more difficut for the linguistically simple reason that the constituents of the first sentence are in the usual mathematical processing order $(10 - 4 = ?)$, whereas in the second they are not $(-4\ [+10] = ?)$. For processing the second sentence its constituents would probably have to be transformed into the usual (or canonical) order to be properly understood. The sentence might be even more difficult if the question part were transposed to the first part of the sentence, thus:

How many bananas do I have left if I take away four bananas from ten bananas?

On the other hand, signaling in the first part of the sentence that a subtraction operator is to follow might make it easier to comprehend than if the subtraction operation were not so indicated, as in:

How many bananas are there if I take away four bananas from ten bananas?

These examples are meant to illustrate how rather simple changes in sentence order may affect comprehension. (They are not, however, descriptions of how they are actually comprehended.) Why sentence comprehension is related to order, even in the sentence cited, is not as easy to explain as it might appear. Accounting for the relative difficulty of the above sentences is a problem for both psycholinguistics (i.e., psychological theories of linguistic performance) and linguistic theory (abstract theories of grammar).[7] The debate in recent years over linguistic theory has been of great significance in its effect on a number of intellectual disciplines including the social sciences, the natural sciences, the humanities, and philosophy. The details of the debate need not concern us here, but one important point of focus has been the

The problem arises in these sentences because the so-called surface structure of sentences (the sentence read or heard), even when the same constituents are contained in them, do not necessarily convey the same (deep structure) meaning. Thus, the different linguistic forms illustrated may have different structure meanings. This phenomenon has now been extensively studied and debated in natural language, but what is not known is the extent to which the same phenomenon holds with mathematical statements. For example, the sentence they are hurting people has two meanings related to two different deep (syntactic) structures depending on whether hurting people is a noun phrase or hurting is a verb. Or, consider the difference between the blind venetian and the venetian blind which is somewhat more closely related to mathematical examples cited. The words are the same in both, and only their word order differs. But, it is clear that there is a considerable difference in meaning. Can one consider the differences in word order in the mathematical sentences cited as involving the same linguistic rules and their consequences for meaning as natural language statement, or do mathematical statements constitute a different class of linguistic objects with different properties. Very little is known of this.

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[7] While this problem has been studied by mathematics educators it is my impression that variations in sentence form that have been employed have not been related to what is known in modern linguistics of the nature of syntactic structure and semantics.

place of "meaning" or "semantic interpretation" in a linguistic system.[8] Psychologists, in addition, have been concerned with the relation of language to thought. It would appear from our discussion of linguistic models for mathematics learning that parallel issues exist in respect to the influence of linguistic form on mathematical reasoning. If it is the case, as in the examples cited, that linguistic form affects understanding, then for instructional purposes, particularly with young learners, the form of communicating mathematical ideas is important. If one wishes learners to understand all forms of mathematical expression, then the nature of the linguistic translation from one to another sentence form has to be known as well, although this does not solve the problem of how these translations and the sentence forms come to be known and understood.

Application of the model. What then are the questions concerning mathematics learning that the language model can illuminate? The most general and at the same time probably the most difficult is whether mathematical reasoning and learning occur according to a linguistic model (i.e., thought = language) or to a linguistic *representational* model (i.e., thought ≠ language). In each case, a particular linguistic theory is required of generative transformational theory (e.g., Chomsky, 1965) that makes distinctions between deep and surface structures and provides for the transformations between them, as well as assigns different roles to syntactic, semantic and phonological components of the grammar (to cite the most prominent of present-day linguistic theories). If language, on the other hand, is considered simply as a representational system for thought, one would wish to know first how linguistic representation facilitates and/or inhibits mathematical reasoning (as in the banana examples), and second, how various linguistic forms represent mathematical ideas.

It is not too well known at present how the lexical features of mathematical language develop and how such knowledge enters into mathematical reasoning and problem solving. One knows little, too, of the reverse, that is, how cognitive structure influences or affects knowledge of the lexicon, although there is increasing research on this question (see Beilin, 1975). It is not at present clear whether mathematical languages have a syntax comparable to natural language syntax or whether other features of the language serve the ordinary functions of syntax. Again, it is not known in what way syntactic structures of mathematical languages, if they exist, relate to the processes of mathematical reasoning and problem solving, whether they are isomorphic to cognitive structures, or whether they are different in kind and simply map onto cognitive structures.

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[8] For those interested in contemporary linguistics, a sophisticated selection of papers is to be found in: Akmajian and Heny (1975), Steinberg and Jackbovits (1971); see also Chomsky (1965, 1968), Fillmore (1968), and McCawley (1968). Good discussions of language development, as well as general issues, are to be found in Brown (1973), Dale (1972), McNeill (1970), and Slobin (1971).

Other processes enter into mathematical reasoning and learning and problem solving that relate significantly to the linguistic functions of mathematics. These are memory processes and imagery. Much of contemporary research into the nature of language comprehension concerns memory for linguistic structure and linguistic meaning (see, for example, Kintsch, 1974). One may similarly concern himself with the role of memory in mathematical reasoning and in the comprehension of mathematical expressions. Will a child, for example, better retain mathematical facts or relations if they are put into natural language forms (sentences) than in special language constructions (formulas, equations, etc.) or the reverse; might it differ for children of different ages? Is there memory for meaning in mathematics that differs from memory for linguistic form? How important is memory of form in contrast to meaning in mathematical reasoning?

Mathematical imagery is said to be particularly evident in the reasoning of the geometer and the learner of geometry. What function does it serve? Is it necessary to geometric reasoning, if not necessary in what ways is it helpful and why? What is the relation between mathematical imagery and mathematical logic?

Does geometric intuition result from geometric imagery, does it have its origin in linguistic form, or does it come from some other source? Is imagery a significant mode of representation in mathematical systems other than geometry (such as algebra)? What relation does it have to linguistic representation in all of mathematics? In "pure" geometry if points, lines, and planes are abstract entities, what are points, lines, and planes in geometric imagery? Are they abstractions, or are they particulars? If they are particulars, how are they translated into the understanding of geometric abstractions that are necessary to the geometer's task?

In sum, there are many aspects of mathematical reasoning and learning that are related to the nature of the system in which mathematical relations are represented and processed. The form and functions of that system provide a model for understanding the nature of mathematics learning; according to some, it provides the only model to account for mathematical reasoning although this interpretation may be too extreme to be probable.

The Cognitive Model of Mathematics Learning

The models above are derived primarily from the properties of mathematics itself, from mathematical logic, and mathematical language. The cognitive model, on the other hand, is based on assumptions about the person who processes information from the world or creates and constructs such knowledge. The ability to construct mathematical theorems and solve mathematical problems is assumed to require certain structures or processing systems. In this model, the emphasis is on the nature of the learner, whether one assumes that mathematics is a unitary or complex system of theories, and whether mathematics is conceived as a hypothetical system or a theory of reality. If one assumes that mathematics is a multiple entity, one can expect an interaction between the type of mathematics (e.g., algebra or geometry) and various forms of cognition, assuming that cognition encompasses more than one kind of process (e.g., imagery, memory, reasoning, etc.).

Developmental models. Two types of cognitive theory provide models of mathematics learning, developmental and nondevelopmental. Developmental theories assume that the cognitive system undergoes change over time, with some theories emphasizing maturational control of behavior, others experiential, and some (like Piaget's) emphasizing the interaction of both maturational and environmental influence. One group of theories assumes further that the changes are stage-like, reflecting qualitative differences in cognitive structure and performance, while another set of developmental theories is based on assumptions of continuity, with changes in performance attributed to units added through experience. The stage theories are best represented by the developmental theory of Piaget, and the continuity theories best represented by Gagné's neobehaviorist theory of cognitive learning. According to the Piagetian model, learning is a function of development, while in Gagné's and similar empiricist theories, development is a function of learning. What is meant by "cognition" is different in each case. Although each theory attempts to account for cognitive processes (or in the case of the behaviorists, for cognitive performance or behavior), Piaget's explanation is based on a structuralist model while Gagné's nonstructuralist theory derives from an associationist model. The difference is important to how learning is conceptualized. Piaget's account embraces the view that structures or schemes are constructed in the course of development from encounters with the real world in which existing cognitive structures interact with inferences (or new schemes) developed from experience. The need to resolve differences between what is known from existing structure (e.g., continuous length) and what is newly experienced (e.g., partitioning) leads to the elaboration of new structures (i.e., of measurement) that incorporate and integrate the elements of the new experience with available structures. Emphasis in Piaget's theory is placed on the dominant role of schemes and cognitive structures. The structures that have a bearing on mathematical reasoning are logical structures, and Piaget is at pains to demonstrate that these logical structures are differently constructed from empirical generalizations or inferences made from physical experience. As a consequence, Piaget proposes the existence of two types of knowledge and two types of internal processing systems. One process, probably that of abstraction and inductive inference, leads to physical knowledge, such as knowledge of color and forms of objects. The other type of knowledge is logico-mathematical and involves processes of deduction that establish relations among concepts achieved by abstraction and inductive inference. Knowledge of the transitivity of weights, for example, is a different kind of knowledge from knowledge that an object has weight. The transitive relation among weights (A>B, B>C, therefore A>C) is a logical relation, and knowledge of it is achieved through a logico-deductive process; whereas knowledge of weight is achieved by inference from experience, that is, from directly holding objects in one's hands. To determine that one object weighs more than another is also empirically determinable. However, to determine whether one object weighs more than another without a direct comparison between them, but only by reference to a third object against which each is compared, requires a purely logical process that does not depend on knowledge of the world for its verification. Such logical knowledge is acquired within a framework of group structures that has an analogue in the group structures of logic and mathematics. Mathematical reasoning and logical thought, according to Piaget, is defined at least in part by the logic of classes, the logic of relations, propositional logic, etc. The ability to reason mathematically is attributed to the development of cognitive systems that are analogues of logical-mathematical systems.

A model based on Gagné's and similar views rests on the assumption that internal "organization" is not in the form of schemes or structures in the sense meant by structuralists like Piaget, but on associational chains. Knowledge gained from new experience becomes associatively linked with old knowledge. These chains might be quite complex and need not be continuous; they might in fact assume the form of the tree structures of classification systems. The learning is conceived as taking place through the cumulative addition of units of experience or knowledge, and no distinction is made between logico-mathematical knowledge and physical knowledge. All knowledge is essentially the same, except for differences in complexity. More complex units are simpler units tied together. Transitivity, for example, would require prior experience or training in comparisons between pairs of constituent elements (e.g., A & B, B & C, C & D etc.) and then training or experience with A & C, B & D, with feedback as to the correctness or incorrectness of response. Learning is the consequence of such experience and not from any conflict between internal structures as in Piaget's theory.

In Piagetian training, there are also experiences with constituents, but the hypothesized change in knowledge (or structures) comes from the encounter between the child's prediction of the relationship between elements (correct or incorrect) and information obtained from directly weighing the elements A & C (correct). Transitivity would be constructed out of conflict between inferences based on old knowledge evident from the child's prediction and the new knowledge obtained [illegible] the weighing. The newly developed structure from the synthesis (transitivity schemes) would enable the child to solve the problem correctly.

The state of the learner in Gagné's model is assumed to be a function of the learned hierarchy of skills acquired through experience; whereas in the Piaget model it is a function of the stage of structural organization. In each model it is assumed that the developmental status of the learner is a significant determinant of his ability to learn.

Nondevelopmental cognitive models. Nondevelopmental models, in general, make the assumption that cognitive processes or structures (or associational organizations) do not undergo developmental change. Such processes are either natively given or develop at such an early age that the systems are instated at the time that the first cognitions may be said to appear. The prototypic theory of this class is information-processing theory.

Information processing languages in contrast to processing theories are formal systems that are used in pure mathematics as a means of representation for Turing machine theory, recursive function theory, and automaton theory. They are used in applied mathematics, in linguistics, computer science, and cognitive psychology (Simon & Newell, 1974). These languages provide a valuable means for the construction of models of the nature of cognition and its functions in concept formation, problem solving, pattern recognition, linguistic processing, etc.

Information processing theories, as the name applies, are based on the thesis that the input to a psychological processing system, which may be an external or internal "stimulus," provides information that is transformed and acted upon in a variety of ways demanded by the task, with the output translated into a verbal, motor or other response, or else stored for future use. The varied types of input information are represented in the system in a consistent form by a "coding device." The system properties define the code. In some models the code is a linguistic code; in others it is a system of patterns or images. It may in fact be any form of symbolic representation. The thus-represented information is processed further depending on the nature of the device. If the task is that of recognition or identification, there may be a match process between incoming coded information and already coded data in the form of templates or otherwise stored information. The results of the match or mismatch may be processed further, so that if a verification of truth value is required, a "truth-index" may be posited in the system and a true-false decision may be made. The thus-processed coded information is then decoded and transformed into some type of response. Characteristically, the system includes feedback procedures or loops whereby output information is recycled and introduced into another processing procedure if it did not in the first place lead to a satisfactory solution. The feedback system permits the system to be self-regulating. The number of processing components and their hypothesized function is defined by the nature of the task and the nature of the information to be processed.

Some information processing theories encompass in their basic formalism set-theoretic and relational concepts (e.g., Reitman, 1965) and others equivalent graph theory. Still others embody a truth-table logic as a basis for decision rules (Bourne, Ekstrand, & Dominowski, 1971). Among the things information processing models attempt to do is explore the manner in which algorithms (systematic solution procedures) and heuristics (procedures for limiting search) enter into problem solving. They test search and scanning schemes which determine the manner in which subgoals are defined and alternative directions of search are scanned (Bourne et al., 1971). Thus, the information processing approach, which is intimately associated with computer processing models in the simulation of intellectual functions, is particularly oriented to the analysis of problem solving strategies. As a simulation model, it is also utilized in exploring the nature of cognitive processing especially in concept learning, game playing (e.g., chess), and language processing contexts.

Applications of the cognitive model. From the description of the respective developmental and nondevelopmental models, it is clear that their aims are in part different. The developmental models, particularly Piaget's, assume that cognitive stages reflect different levels of cognitive structure and propose that the ability to learn particular logical or mathematical tasks is a function of the child's cognitive level. The behavioristic cognitive models assume the existence of skill hierarchies, and learning is a function of the developmental (i.e., experiential) achievements of these skills. Thus, one of the principal issues that has to be decided in respect to mathematics learning is whether such learning is under control of development or the reverse, that development is a function of learning. A whole series of individual questions can be asked concerning the relation between specific hypothesized cognitive structures (in Piaget's theory) and the acquisition of specific mathematical concepts. What, for example, are the cognitive structures necessary to an understanding of the tenets of Lobachevskian and other non-Euclidean geometries? Are these cognitive structures the same as required for understanding Euclidean geometry? If not, what are the differences, and how do they arise? In addition, must one kind of knowledge or cognitive structure be acquired prior to another before certain types of mathematics can be learned? Behavioristic theories assume only that the constituents be known before the more complex system can be constituted into a larger unit.

The cognitive model as already suggested intersects with the logical and linguistic models. The extent to which logic, cognition and linguistic theories, and the processes to which they refer, are interrelated is as yet little known. Mathematical learning provides a natural context in which to study these interrelations.

The relation of curriculum design to cognitive development is often alluded to, yet is studied relatively little. Realistic efforts to develop articulated curricula according to the cognitive model which are joined to a program of experimentation on learning would appear to be long overdue. Maybe it is so long overdue that it is no longer worthwhile doing. Whether it is or not, the problems and questions related to mathematics curriculum development remain. Contemporary interest in particular curriculum designs seems to be a function more of political, economic, and social commitments than to scientific decision making bearing on what contributes most effectively to learning and knowledge acquisition.

Insights into cognition and problem solving provided by nondevelopmental models offer added tools to the exploration of mathematical learning and reasoning. While these approaches have been undoubtedly oversold by premature large-scale application through programmed instructional methods, the computer and information processing models still offer promising approaches to understanding the nature of problem solving. If they have not as yet fully exposed the properties of problem solving and other forms of reasoning, they have added much to what had been known. Whether mathematical problem solving has benefited from this knowledge would require a hazardous guess, but the road would seem open to a great deal more research of this kind.

Understanding Mathematics Learning

What I propose is that an understanding of the processes by which knowledge of mathematics is achieved requires the application of each of the foregoing models. No full comprehension of the interaction between developing cognition and the complex fields of mathematics is likely without a conceptualization of both mathematics as a set of logical, linguistic, and computational theories and the learner as a complex of developing cognitive structures and processes.

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The Erlanger Programm As a Model of the Child's Construction of Space

J. Larry Martin

Missouri Southern State College

Many mathematicians perceive beauty in the precision of mathematical structures. For educators who may choose to seek erotica in other "forms," there is still reason to consider mathematical structure. Shulman (1970, p.22) has stated that,

> to detemine whether a child is ready to learn a particular concept or principle, one analyzes the structure of that to be taught and compares it with what is already known about the cognitive structure of the child.

Smock in this monograph has emphasized that the learning environment must be considered from two frames of reference: the operational systems determining the child's interpretation of environmental events and the inherent sequence and structure of the content. Since we are considering the child's conception of space, it is natural to examine the structure of geometry. It is important to note that the domain of discussion is space, not geometry. Geometry will be used to provide models of the child's conception of space.

The Erlanger Programm

In a lecture in 1872, Felix Kelin presented his now famous definition of a geometry: "A geometry is the study of those properties of a set X which remain invariant when the elements of X are subject to the transformations of some transformation group" (Tuller, 1967, p. 70). Some underlying concepts are needed to understand this definition. A few more basic definitions are provided first. A more careful analysis of the meaning and implications of the Programm follows throughout the paper.

Ordinarily, a person regards a transformation as a change. In mathematics, a transformation may be regarded as a rule associating points of a set X with points of a set Y. More explicitly, a transformation of the set X into the set Y can be thought of as a rule of correspondence that assigns to each element of set X one and only one element of a set Y. If y in Y is associated with x in X, then y is called the image of x. Denoting the transformation by f, "y is the image of x" can be symbolized $y = f(x)$. If every element of Y is assigned to some element of X, the transformation is called a transformation of X onto Y. If no element in Y is the image of more than one element of X, the transformation is said to be one-to-one.

The transformations of the Erlanger Programm are from a set X onto the same set X. That is, they assign to each element of the set X an element of the same set X. If such a set T of transformations of X onto X have the following two properties the set T of transformations may be properly called a group of transformations or transformation group:

1. The inverse of every transformation in T is itself a transformation in T.

2. The resultant of any two transformations (distinct or not) in T is also a transformation in T. That is, if f_1 and f_2 are transformations in T, then there exists a third transformation f_3 in T which has the same effect on X as do f_1 and f_2 applied successively.

A property of a set X which is unchanged under all the transformations of the group is called an invariant property of set X under that transformation group. If a subset S of a group T of transformations itself forms a group, then S is called a subgroup of T.

The remainder of this paper will discuss the Erlanger Programm and its applicability as a model of the child's conception of space. The paper is organized around consideration of the questions:

1. What are the transformations?

2. What are the invariants?

3. What is the set X?

4. What are the subgroup relationships existing among the various geometries?

5. What is the nature of spatial reality?

6. What are the consequences for research of appealing to the Erlanger Programm as a model of the child's conception of space?

An attempt will be made to point out that many of these questions have alternative answers. When selecting one set of answers to the questions, one should at least be aware of the fact that he has made one choice and discarded others.

Transformations and Invariants

Using Klein's definition of a geometry, it is possible to categorize and name various geometries according to both the transformation groups involved and the invariant properties under that group. Figure 1 shows the relationships existing among some transformations groups (geometries).

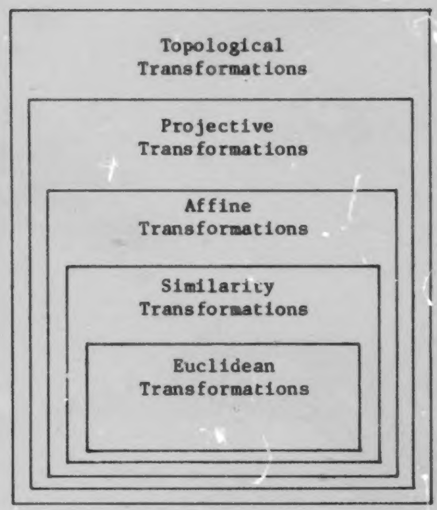
Topological Transformations
Projective Transformations
Affine Transformations
Similarity Transformations
Euclidean Transformations

Figure 1. A hierarchy of transformation subgroups.

This hierarchy indicates that if certain stipulations are made (these stipulations are discussed later), then the group of Euclidean transformations is a subgroup of the group of similarity transformations, the similarity group is a subgroup of the group of affine transformations, the affine group is a subgroup of the group of projective transformations, and the projective group is a subgroup of the group of topological transformations.

The transformations of topology are called homeomorphisms. A one-to-one transformation f from X onto Y is called a homeomorphism if it is continuous and reversibly continuous. That is, f is one-to-one, onto and continuous and the inverse of f is also continuous. Roughly speaking, then, topology is the study of properties that remain invariant under homeomorphisms. These invariants are called topological properties. An intuitive notion of homeomorphism will help identify some of these properties. If one figure can be distorted into a second figure by no more than pulling, bending, stretching, or shrinking, then the two figures are topologically equivalent or homeomorphic. Shape distortions such as these are always homeomorphisms. There are also homeomorphisms other than shape distortions. However, this intuitive idea of homeomorphisms as shape distortions will suffice for now.

Applying this low-powered concept of topological transformations demonstrates that shape and size are definitely not topological properties. Neither is "straightness." What are some topological properties? Examples are interior of a set, exterior of a set, boundary but not boundedness of a set, connectedness of a set, linear and cylic order, and openness and closedness of curves.

If topology is characterized as the study of invariant properties under the group of homeomorphisms, projective geometry can be characterized as the study of properties invariant under the group of collineations.[1] Collineations are special homeomorphisms which transform collinear points into collinear points and, hence, lines into lines. Concurrence of lines is a projective property. That is, if three or more lines intersect at one point, then the lines resulting from the transformation will also intersect at one point. Also a polygon of n sides will transform into a polygon of n sides. To illustrate, triangles will go into triangles and quadrilaterals into quadrilaterals.

Affine geometry is obtained as a subgeometry of projective geometry by restricting the group of projective transformations in such a way as to introduce parallelism. Besides parallelism, affine invariants are betweenness of points and ratios of distances. Note the introduction of distance. An affine transformation multiplies all distances on the same line or on parallel lines by the same amount, that is, by the same positive constant. Thus, the ratio of two distances on the same line or on parallel lines is preserved. In particular, affine transformations send equal distances into equal distances on the same or parallel lines and midpoints into midpoints.

Whereas an affine transformation multiplies distances in the same direction by a constant, a similarity transformation multiplies all distances by the same positive number K. K is called the ratio of the similarity transformation. Angle measure is a similarity invariant. The shape of a configuration is preserved but not its size.

If the ratio of similarity is one, distance between points is an invariant property. Such similarity transformations are rigid motions or isometries. These are the transformations of Euclidean geometry. The group of Euclidean transformations consists of translations, rotations, reflections, and compositions of translations and reflections which have fixed lines, i.e., a unique line is associated by the composition with itself. These special compositions are called glide-reflections.

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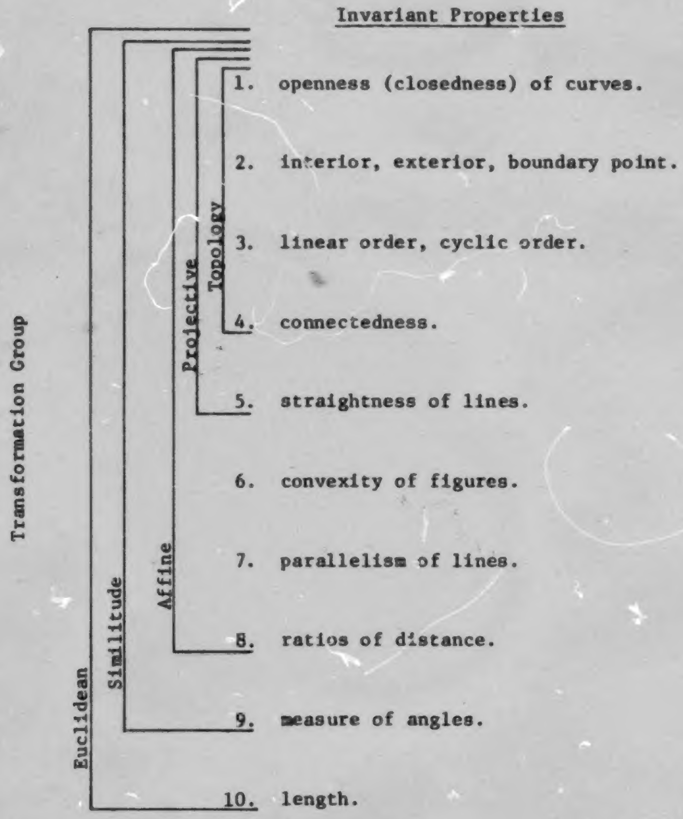
[1] It is possible to define a "general projective group" which includes the group of collineations as a subgroup. See Gans (1969, p. 342) or Tuller (1967, p. 102).

Ideally the properties held invariant by a group of transformations are also invariant under any subgroup. Invariants of the topological transformation group are invariants of the projective transformation group, the affine transformation group, the similarity transformation group, and the Euclidean transformation group. Figure 2 provides an outline of the relationships existing among the invariant properties of the various geometries.



1. openness (closedness) of curves.
2. interior, exterior, boundary point.
3. linear order, cyclic order.
4. connectedness.
5. straightness of lines.
6. convexity of figures.
7. parallelism of lines.
8. ratios of distance.
9. measure of angles.
10. length.

Figure 2. Properties invariant under transformation groups.

The Set X

The word "ideally" was used at the beginning of the previous paragraph because in actuality a subgeometry has the properties of its parent geometry only for the point set which they have in common. Klein talked of the "properties of a set X which remain invariant." Until now the reader has been left to furnish his own set X and interpret the discussion of invariants in terms of this set. As will be shown, the selection of X can have noteworthy effects on a categorization scheme of various geometries.

The discussion of geometries began with the hope that they would provide models of the child's conception of space. As the child constructs the "reality" of space, different geometries might model the child's conception at different stages of his construction. But what is the endpoint of development? As Smock pointed out:

> Analysis of cognitive learning and development, then, is always "biased" by the fact of a context of preconceived ideas of reality...and a particular set of concepts or theory and selected observations.... The designation of a conception of space toward which the child will most likely develop, i.e., that conception held by most adults, is the critical first step. Observations and interpretations of the child's behavior are organized around the specifications inherent in that 'endpoint' of development. (1974, p. 145)

Interpretations, then, about the child's conception of space are based on his progress in the construction of a spatial "reality" with a direction of "progress" and an "endpoint" determined by forces external to the child (e.g., culture, curriculum, adults, etc.). The child's conceptual growth, if not in the prescribed direction of progress, might go unnoticed or misinterpreted.

What is the conception of space held by most adults? One can glibly say that it is flexible. That is, an adult can shift, for example, from projective to Euclidean spatial representation and back again depending upon what the situation and circumstances seem to dictate. But how would the question have been answered a few hundred years ago before the advent of projective geometry? Presumably the space in which a 14th century man moved was exactly the same as the space of modern man. But what of his representational space?

The desire of Renaissance painters to produce a visual geometry provided the impetus for projective geometry. They asked questions like "how can the way things really look be represented in a drawing?" Thus, a geometry of vision developed rather than one of measurement (Gould, 1957, p. 299). The picture made by a painter can be regarded as a projection of objects in space onto canvas. Since length and angles are distorted, how is it that objects are recognizable? It must be that the properties invariant under projection provide the clues.

Let us review the idea of projecting points from one plane in Euclidean space onto another plane. Consider a glass table on which a magazine with corners A, B, C, D is lying as shown in Figure 3. If a lamp is placed at point L not in the plane of the table top nor in the plane of the floor, then the shadow on the floor of the magazine on the table will be as represented by the Figure A', B',C',D'. The projection illustrated is called a central projection of a plane onto a parallel plane. It is a homeomorphism which multiplies all distances by the same constant k, thus preserving shape but not necessarily size. As has already been stated, such a transformation is a similarity transformation.

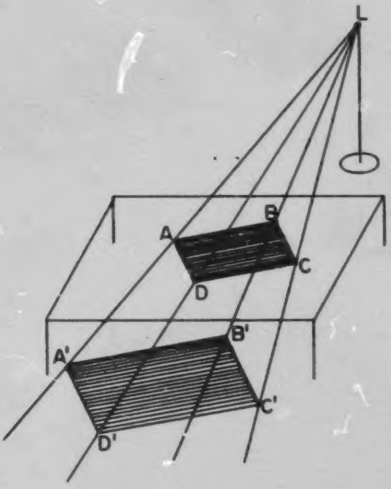


Figure 3. Central projection, parallel planes (Dorwart, 1966, p. 6).

If the table is tipped, the planes are no longer parallel. Such a situation is demonstrated in Figure 4. What would be the effect on the shadow? This obviously creates some problems. Where does such a transformation fit in Figure 2? It does not conserve length, measure of angles, intersection of lines, nor closeness of curves. In fact, it is not even a one-to-one transformation. Point A in plane π has no image in plane π'! Hence, the transformation cannot possibly be a homeomorphism. The set X of Klein's definition of a geometry cannot, then, be ordinary Euclidean space if the classification scheme of Figure 2 is to hold. The Euclidean concept of space must be extended in such a manner so as to eliminate the difficulties noted.

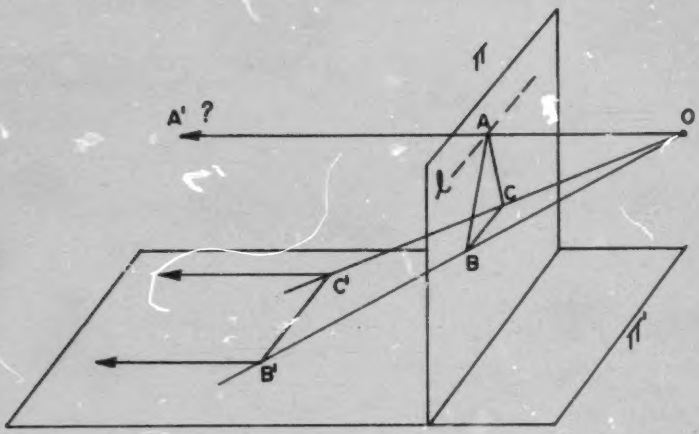


Figure 4. Central projection, nonparallel planes.

In order to portray three-dimensional scenes in two-dimensional drawings, Renaissance painters drew parallel lines as if they actually met in the distance. Since the images in π' (Figure 4) of segments AB and AC in π appear to meet at some distant point, why not, with the artist's "vanishing point" in mind, create a new point to serve as the image of A? Then every point of the triangle ABC in π would have an image in π', and the image of a triangle would be a triangle which is what is desired. This new point is called an ideal point as opposed to the ordinary points of Euclidean space.

A little imagination raises several questions here. Two parallel lines, appear to meet at both "ends." Should two ideal points be added? What about a third line parallel to the two in question? Should more ideal points be added to represent its apparent distant intersections with the two given parallel lines? These questions are dealt with in the following way:

1. To each straight line in Euclidean space a single ideal point is added. The geometrical object resulting is called an extended line.

2. The ideal points which are added to two parallel straight lines are the same.

3. The ideal points which are added to two nonparallel straight lines are distinct.

4. The geometrical object which consists of Euclidean space and all ideal points is called extended space (Gans, 1969, p. 231).

While these agreements may appear arbitrary, they are motivated by two desires. One desire is to preserve an original law in Euclidean space (i.e., through every two points exactly one line may be drawn). Second, the new law should be consistent with the visual geometry of the artist (i.e., every two lines in a plane intersect in exactly one point). Let us examine some of the consequences of these agreements.

To each given ordinary line an ideal point was added. This ideal point also belongs to all ordinary lines parallel to the given line. Thus, lines parallel in Euclidean space will meet at an ideal point in the new space. Different families of parallels will meet at different ideal points. Consequently, any two coplanar extended lines will intersect at exactly one point: an ordinary point if they intersect in Euclidean space, an ideal point if they are parallel in Euclidean space.

It is a characteristic of Euclidean space that two points determine a line. Consider one ideal point and one ordinary point P. They determine an extended line through P in the direction determined by the ideal point. If two ideal points are chosen, what unique line is determined? It cannot be an extended line because extended lines contain only one ideal point. Moreover, it cannot contain any ordinary points because an ordinary point and an ideal point determine an extended line. Logically it must consist of only ideal points. Just as each line in Euclidean space was extended to include an ideal point, each plane in Euclidean space will be extended to include one ideal line. An ideal line is composed of all the ideal points associated with the lines in a given plane. It naturally intersects any other line in its plane at an ideal point. Thus, any two lines in an extended plane, whether two extended lines or one extended and one ideal, intersect in exactly one point.

So far ideal points have been added to produce extended lines. In turn new lines, ideal lines, were created. For each plane there is an ideal line making an extended plane. The situation in three dimensions is similar. Ordinary three-dimensional Euclidean space is extended by the addition of an ideal plane which consists of all the ideal points. All the ideal lines lie in this ideal plane. Two extended planes associated with parallel planes in Euclidean three-dimensional space meet in an ideal line, and two extended planes not associated with parallel planes in Euclidean space meet in an extended line. Similarly, the ideal plane meets each extended plane in an ideal line. Therefore, in this extended space, any two planes meet in a unique line.

To summarize, new points called ideal points have been added to Euclidean three-dimensional space, resulting in extended lines, some completely new lines called ideal lines, extended planes, and one completely new plane called the ideal plane. The new system of points, lines, and planes is extended Euclidean space or real projective space.

Consider Figure 5 as representing sets of points in real projective space. Figure 5a represents two lines which intersect at an ideal point, the same ideal point "at both ends." Figure 5b represents a triangle with one vertex at an ideal point. Returning to Figure 4, the image of triangle ABC will be triangle A'B'C' where A' is the ideal point associated with the extended line through O and A. The transformation from π to π' now preserves closeness of curves. Intersecting lines have intersecting images. The image of line *l* is an ideal line, so lines transform into lines.

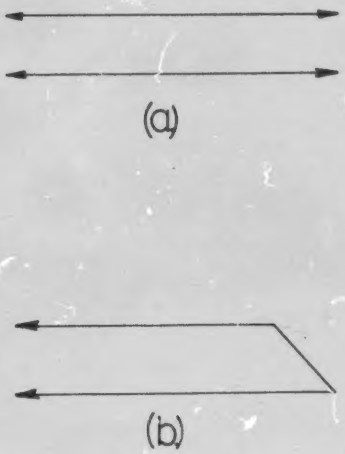


Figure 5. Intersecting Lines (a) and a triangle (b) in Real Projective Space.

Calling these new mathematical entities points, lines, and planes may cause consternation for some. After all, how do you locate an ideal point in space? How can you draw an ideal line? Where is the ideal plane? One needs to recall the dialogue between Alice and Humpty Dumpty in *Through the Looking-Glass:*

> "When I use a word," Humpty Dumpty said in rather a scornful tone, "it means just what I choose it to mean - neither more nor less." "The question is," said Alice, "whether you can make words mean so many different things." "The question is," said Humpty Dumpty, "which is to be master-that's all."

Hierarchies of Geometries

Using extended Euclidean space for the set X of Klein's definition of a geometry, the projective transformations are one-to-one transformations. But before projective transformations can be considered a subgroup of the topological transformations, projective transformations must be continuous with continuous inverses. What does it mean for a transformation to be continuous? One definition is as follows. Suppose f is a transformation from a set X to set Y. Then f is said to be continuous at a point x in X if for any distance e > 0, no matter how small, there exists a distance d > 0 such that whenever a point p in X is within a distance d of x, the image of p in Y will be within a distance e of the image of x in Y. A transformation is said to be continuous if it is continuous at each point in X. Loosely speaking, points close together in X have images close together in Y.

However, since projective geometry is ordinarily considered to be of a nonmetric character, it would seem that the definition of continuity should not involve notions of distance. First, consider a neighborhood of a point. In topological spaces, neighborhoods are sets satisfying certain specified conditions. This paper will not delineate these conditions. In the Euclidean plane, a basic neighborhood of a point is the interior of a circle containing that point. In Euclidean three-dimensional space, a basic neighborhood of a point is the interior of a sphere containing the point. Thus, each point in a topological space can have many neighborhoods, and a set may be a neighborhood of many points. In Euclidean space each point has many neighborhoods, some large and some small. Although the usual neighborhoods in Euclidean space involve the notion of distance, in general topological spaces distance need not be involved. For instance, a neighborhood of a point could be the interior of any region formed by a simple closed curve surrounding the point.

Continuity can be defined in terms of neighborhoods. A transformation f of a set X into a set Y is said to be continuous at the point x in X if for each neighborhood U of the image of x in Y, there is a neighborhood V of x in X such that the image of V is contained in U. The transformation is continuous if it is continuous at each point in X. For example, if the sets X and Y were ordinary Euclidean planes, a function f from X to Y would be continuous at x in X if for any circle C around the image of x in Y there exists a circle V around x whose image was contained in U, that is, V was a subset of U. The situation is demonstrated in Figure 6.

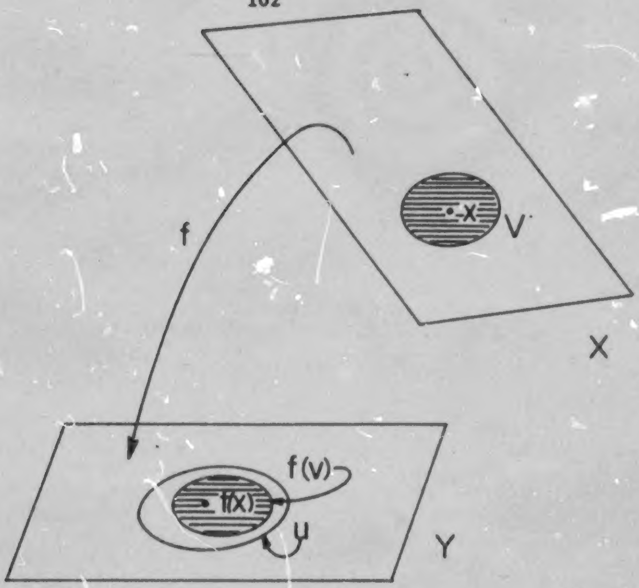


Figure 6. A continuous function.

Using this definition of continuity, projective transformations can be regarded as homeomorphisms without using distance, and the projective group can be regarded as a subgroup of the topological group. In what way can the affine group be considered a subgroup of the projective group? Parallelism is an affine invariant. Yet in extended Euclidean space any two coplanar lines meet. It would seem that a subgeometry should have the properties of its parent geometry. To eliminate the conflict, one could define two lines as parallel if they meet at an ideal point. Other agreements would need to be made about ideal lines. An alternate approach would be to simply restrict the set X to the ordinary points of three-dimensional space. Then two coplanar lines are parallel if they have no points in common. The latter approach is adopted here.

The statement "a subgeometry has the properties of its parent geometry only for the point set which they have in common" was used earlier in this paper. If the set X is restricted to ordinary points, then the affine transformations form a subgroup of the projective transformations. For this restriction on X, properties invariant under projective transformations will remain invariant under affine transformations. For example, intersecting (at ordinary points) lines will have intersecting images. In this restriction of extended Euclidean space to ordinary Euclidean space, the similarity transformations form a subgroup of the affine transformations and the Euclidean transformations form a subgroup of the similarity transformations. It is within this context that the classification of geometries presented in Figure 2 is valid.

Alternative structures - Euclidean. The nesting of the transformation groups as displayed in Figure 2 is dependent upon the set X, that is, the space upon which the transformations act. Ideal points were added to ordinary Euclidean space so that this nesting would hold. What would be a classification of these geometries if ordinary space were not extended? It has been shown that in ordinary Euclidean space projections need not be one-to-one, and thus, not homeomorphisms. Projections are not a subgroup of topological transformations in ordinary space. However, extended space was restricted to ordinary Euclidean space for affine transformations. If the set X is ordinary Euclidean space for all the transformation groups mentioned, the subgroup relationships are as shown in Figure 7. Affine geometry is a subgeometry of topology and of projective geometry, but topology and projective geometry are on separate branches of the classification scheme.

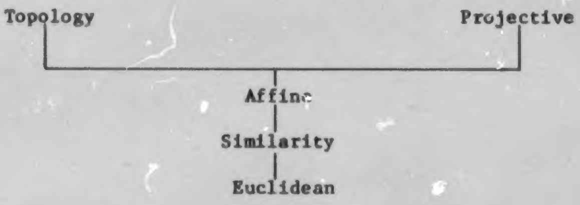


Figure 7. A classification of geometries in Euclidean space.

Whether the set X is ordinary or extended space, there are other geometries that could be placed in a classification scheme based on subgroup relationships. For example, the similarity transformations are a shape preserving subgroup of the affine transformations. There is another subgroup of the affine group that preserves area but not necessarily shape. This group is called the equiareal or equiaffine group of transformations. A classification scheme could include the equiareal group as shown in Figure 8.

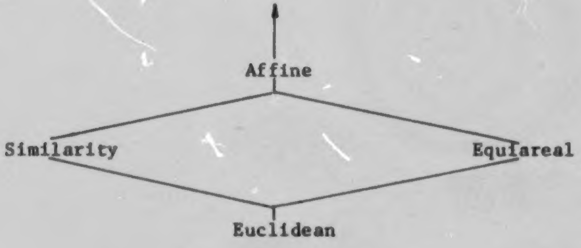


Figure 8. A classification including equiareal transformations.

Alternative structure - non-Euclidean. Thus far subgroups of the topological group have been considered only according to classification schemes culminating with Euclidean geometry. Other endpoints are possible. Suppose a specialized conic, for example an ellipse, is chosen in the real projective plane (extended Euclidean plane). Call the points interior to the ellipse ordinary, points on the ellipse ideal, and points exterior to the ellipse ultraideal. If the points on the ellipse and the exterior points are deleted from the real projective plane, what is left is called a *hyperbolic plane*. This presents another set X to which Klein's definition of a geometry can be applied. The transformation group consists of the collineations that send points on the ellipse into points on the ellipse and send interior points into interior points. Each of these transformations is a projective transformation, but they behave in the special way described. The study of the invariants of this transformation group with the hyperbolic plane taken as the set X is called *hyperbolic geometry*. Since these transformations are collineations, hence projections, hyperbolic geometry is a subgeometry of projective geometry.

Klein presented a model, shown in Figure 9, of the hyperbolic plane. The "plane" consists of only the points interior to the ellipse. The "lines" of the plane are chords of the ellipse. Lines are "parallel" if they meet at an ideal point. Lines are "nonintersecting" if they are part of projective lines which meet at an ultraideal point. "Nonintersecting" and "parallel" are not synonyms. From the nature of the particular collineations forming the hyperbolic group (i.e., collineations send ideal points to ideal points and ordinary points to ordinary points), it follows that parallel lines will have parallel images. Also nonintersecting lines will have nonintersecting images. That is, parallelism and non-intersection are hyperbolic invariants. Some striking characteristics of this geometry are: (a) the sum of the measures of the angles in a triangle is less than a straight angle, (b) there exist lines parallel to both of a pair of intersecting lines, (c) given a line and a point not on the line there exist exactly two lines through the given point parallel to the given line, and (d) given a line and a point not on the line there exist infinitely many lines through the given point which do not intersect the given line. The last three of these situations are displayed in Figure 10.

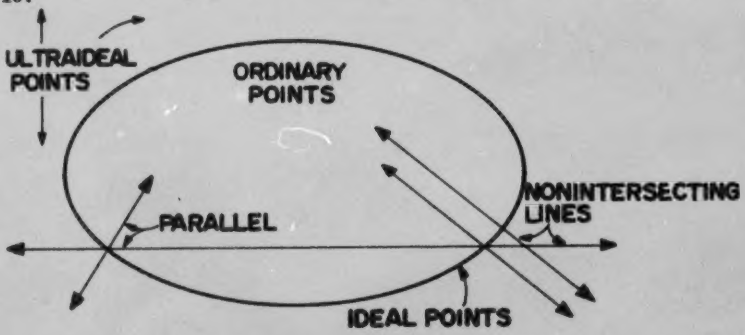


Figure 9. Klein's model of the hyperbolic plane (Fishback, 1969, p. 229).

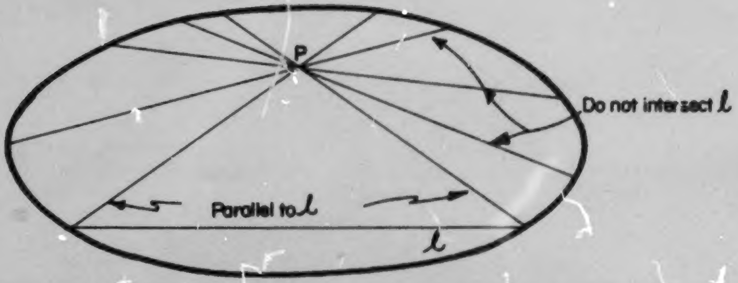

P
Do not intersect ℓ
Parallel to ℓ
ℓ

Figure 10. Some characteristics of hyperbolic geometry.

There are other non-Euclidean geometries. If the space X is taken to be the surface of a sphere and straight lines are defined as great circles on the sphere, two lines always intersect. In fact, they intersect twice. No parallels would exist for a given line. This would seem to be close to the "reality" of a navigator on the earth. The transformations of this space form a subgroup of the projective group. The geometry is called *spherical geometry* or *double elliptic geometry*. In this "reality," the sum of the measures or the angles in a triangle is more than a straight angle. Another non-Euclidean subgeometry of projective geometry is *single elliptic* or merely *elliptic* geometry. In this geometry, two lines always meet in exactly one point and enclose an area. The space X of this geometry is similar to real projective space. Thus there are many possible "paths of progress" and "endpoints" for spatial conceptualization suggested by an analysis of geometries via Klein's definition. Some are Euclidean; some are non-Euclidean. Even Euclidean "endpoints" could be arrived at by following different "paths of progress." The relationship of the non-Euclidean geometries to those geometries culminating in Euclidean geometry is shown in Figure 11.

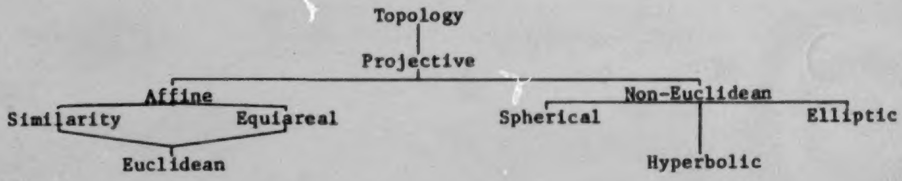

Topology
Projective
Affine
Non-Euclidean
Similarity
Equiareal
Spherical
Elliptic
Euclidean
Hyperbolic

Figure 11. A classification including non-Euclidean geometries.

Spatial Reality

The Physical Reality of Space

Klein's definition of a geometry has been used to identify and classify many geometries. Perhaps these geometries can provide models for the study of the child's construction of space. Questions consequent to such an attempted modeling of the child's construction are discussed later in the paper. Before these questions and before examining the nature of the child's conception of space, it would seem natural to examine the prototype, that is, the physical reality of space.

Some may consider geometries other than Euclidean to be strictly formal, of interest only as an intellectual exercise. They might accept that they are logically developed and internally consistent while still rejecting their "truth." They would view axioms of such systems to be arbitrary statements and the concepts to be merely symbols with which to operate. That a non-Euclidean geometry could have any correspondence to physical reality was considered absurd 200 years ago. Kant, one of the most influential philosophers of the late 1700's, held as a basic tenet that "Euclid's axioms are inherent in the human mind, and therefore have an objective validity for 'real' space" (Courant & Robbins, 1961, p. 219). However, Klein (1939) points out that "our space perception is adapted only to a limited part of space, and then only with a limited degree of accuracy and can be satistifed by either hyperbolic or spherical geometry" (p. 179).

The question as to which geometry should be preferred as a model of the physical world was raised long before Klein made the statement included in the preceding paragraph. Gauss reportedly[2] attempted to settle the question by measuring the angles in a triangle whose vertices were the peaks of three mountains about 100 miles apart. The sum of the angles was not sufficiently different from 180° to suggest a non-Euclidean geometry. Had the sum been noticeably less than 180°, hyperbolic geometry might have been preferable to describe physical reality. However, the variation from 180° was small enough to fall within the error of measurement.

Lobachevski experimented on a larger scale. Using a fixed star and positions in the earth's orbit six months apart, he concluded that to find a measureable defect from 180°, one would need to use a triangle with sides many million times as great as the distance from the earth to the sun (Gould, 1957, p. 294). Thus, both experiments were inconclusive. But they did demonstrate that for distances of a few million miles either Euclidean or non-Euclidean geometry can serve as a model.

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[2] While Gauss did measure these angles, there is some doubt that his purpose was to check which geometry was most appropriate to physical reality. See Boyer (1964) and Gauss (1880).

In the early 20th century, Poincaré pointed out that physical experiments must start with certain axioms about physical reality just as the geometer starts with axioms for his geometry. If the physical definition of "straight line" is the path of a ray of light, then the mathematician must take this into account as he tests a geometrical model against physical reality. Geometrical properties of straight lines defined as paths of light rays could differ from those of Euclidean straight lines. With this in mind, suppose Gauss had obtained less than 180° as the sum of the angles in the triangle formed by the mountain peaks. This could be explained by a hyperbolic space. Or it could be that space is Euclidean but that light rays travel in a curved path and not in the straight lines of Euclidean geometry (see Figure 12). The discrepancy would be due to two different meanings of straight line: the physicist's and the geometer's. Thus, different systems of geometry can describe the same physical reality if the axioms of physics are altered.

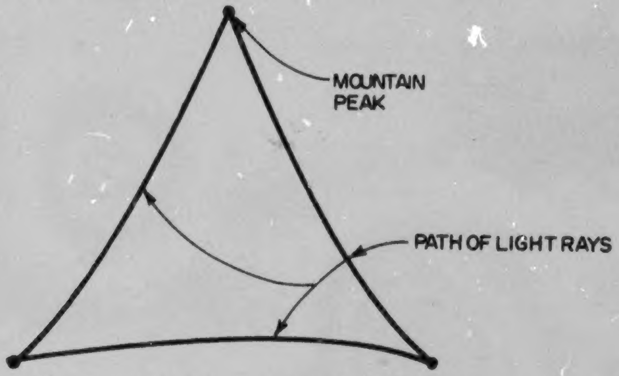

Figure 12. An alternative to hyperbolic space: curved light rays in Euclidean space.

Einstein's theory of relativity utilizes a curved space. The navigator uses spherical geometry. Recent research in optics suggests that three-dimensional hyperbolic geometry can be used as a model for visual space (Blank, 1958; Blank, 1961; Luneburg, 1950). Poincaré stated "one geometry cannot be more true than another; it can only be more convenient" (Coxeter, 1969, p. 288). If the choice of a geometry is merely one of simplicity or convenience, Euclidean geometry would seem the best choice to many.

However, convenience is not the central concern. It must be kept in mind that it is the child's construction of his spatial reality that is the primary concern, not the geometry of physical reality or the child's conception of hyperbolic, projective, Euclidean, or any other specific system of geometry.

The Child's Spatial Reality

Thus far the discussion of spatial reality has dealt only with the nature of physical space, a space external to the child. Whatever geometry one uses to model physical space, this space exists external to the child's construction of his spatial reality. The main objective of this paper is to model the child's construction using geometries from the Erlanger Programm. Hence, attention is now focused upon the child. Since Piaget has described a rather comprehensive theory of the child's conception of space, his theory provides the foundation for much (though there are intentional differences) of the following discussion.

At least five aspects of the child's space warrant the attention of mathematics educators and psychologists: (a) visual space, (b) sensorimotor space, (c) perceptual space, (d) representational space, and (e) conceptual space. The following discussion attempts to clarify the nature of and the relationships existing among these various aspects of the child's space.

Visual space. A distinction must be made between visual space and physical space. There is an immediate visual sensation or experience which is a function of at least the variable factors of time and location of the observer and the invariable factor (at least for limited time spans) of the physiological characteristics of the observer. A geometry modeling these visual sensations (visual space) could be entirely different than a geometry modeling physical space. Visual space is bounded; physical space may not be. In visual space, changes in shape and apparent distances occur; in physical space shape and distance may be invariant. For example, two objects A and B may be placed on a table. As an observer walks around the table, the distance between A and B visually changes. The shapes of A and B are visually different from different vantage points. However, in physical space the distance between A and B does not change nor do A and B change shape.

Optics research (Luneburg, 1950) demonstrates that visual space possesses a uniquely defined metric and that this metric is the metric of three-dimensional hyperbolic geometry. Blank (1958, 1961) analyzes Luneburg's theory and makes explicit its underlying assumptions. He reports several experiments to substantiate Luneburg's claims. While these articles presuppose a certain mathematical sophistication, they need to be interpreted by mathematics educators and psychologists. It would appear obvious that the child's construction of space is heavily influenced by the nature of visual space.

Sensori-motor space. Sensori-motor space begins with a set of unrelated spaces: oral, tactile, postural, visual, and auditory. Each of these spaces is body centered, though the child's own body may not be considered as part of any of these spaces. Toward the end of sensori-motor development these individual spaces are coordinated into a single space in which the body is one object among others. The child develops a concept of object permanence (see Smock, 1975) and objects gain an independence from the child's body (Beth & Piaget, 1966).

Perceptual and representational space. "Perception is the knowledge of objects resulting from direct contact with them" (Piaget & Inhelder, 1967, p. 17). At first, perceptual space is included in sensori-motor space. To use only the visual component of sensori-motor space as an example, before perceptual space differentiates itself, a child's perception of an object may coincide precisely with his visual image of the object (the object in visual space). But perceptual space is constantly enriched by the child's activity. His perceptual space evolves into a synthesis of the knowledge resulting from this activity and his visual space. Objects seen in perspective, for example, can be related to the observer's knowledge of the objects. Thus, what in visual space may be a trapezoid can be perceived (perceptual space) as a square or a rectangle.

Just as perceptual space extends sensori-motor space, representational space extends perceptual space. Representation "involves the evocation of objects in their absence or, when it runs parallel to perception, in their presence. It completes perceptual knowledge by reference to objects not actually perceived" (Piaget & Inhelder, 1967, p. 17). It is one thing to recognize or perceive that two lines are parallel or that two figures are similar. It is quite another to be able to construct a figure similar to an existing model. While extending perception, representation introduces a new element into the child's construction of space, a system of significations. The child now has available to him reconstructable representations or images. These internalized imitations are distinct from perceptions and are recognized as such by the child. At the level of representation the child can differentiate between the symbol and that which is symbolized.

Conceptual space. The evolution of the nature of images provides the basic distinction between representational space and conceptual space. Piaget and Inhelder (1971) have studied the nature of imagery and defined the mental image as the "evocation of a model without direct perception of it" (p. 4). The function of the image is to provide a faithful and accurate copy of the model. While not a direct prolongation of a perception, the image, which is an internalized imitation, can reproduce the content of perception, e.g., shape and color. Naturally the imitation only includes what the child understands and considers typical or exemplary.

According to Piaget and Inhelder (1971), the two main stages in image development correspond to the preoperational and operational stages of the child's cognitive development. Up to the age of about seven or eight years, images are essentially static, direct copy images. While not static in all respects, the preoperational image fails to coordinate states and transformations. The static nature of these images is due primarily to the limitations of preoperational thought. That is, transformations are slighted in favor of states. "Generally speaking preoperational thought may be thought of as a system of notions within which figurative treatment of states takes precedence over comprehension of transformations" (Piaget & Inhelder, 1971, p. 17).

There is a pseudo-conservation peculiar to the preoperational images. Pseudo-conservation is best explained with an example from Piaget and Inhelder's work (1971). Children were shown two cardboard squares (see Figure 13a) and asked to imagine what the figure would look like if the top square were moved slightly to the right (see Figure 13b). Drawings of the youngest children (4 years) tended to show the squares completely separated (see Figure 13c) or put together in a new way (see Figure 13d). But as the children began to imagine the glide, pseudo-conservation became more prevalent (see Figure 13e). Whereas operational conservation would keep the shapes and sizes of the squares constant, preoperational pseudo-conservation attends more to boundaries. Note that the characteristics which the child chooses to leave invariant are precisely those that are modified in actuality. On the other hand, characteristics which he alters are actually invariants. In other tasks pseudo-conservation manifested itself in a reluctance to violate interiors or enclosures when a transformation resulted in intersecting figures.

> Pseudo-conservation arises, then, when a subject retains certain characteristics of an object which he considers typical or exemplary, and which he clings to at the expense of other apparently more important characteristics. (Piaget & Inhelder, 1971, p. 362)

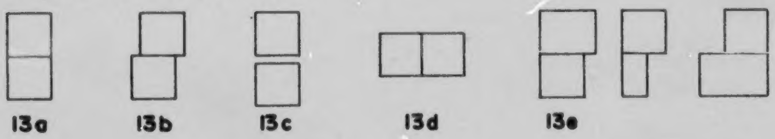
13a 13b 13c 13d 13e

Figure 13

The imagery during the preoperational stage of development utilizes the figurative aspect of thought almost exclusively. As the image is increasingly directed by the child's active operations, the figural aspect becomes more and more subordinated to the operative aspect of thought. With the formation of operations and operative structures, children become capable of thinking in terms of transformations. The mobility of the operations is reflected in their images. Images become more mobile, more anticipatory. Whereas at the preoperational level of development, figurative functions, and imagery in particular, govern thought, the situation is reversed at the operational level. That is, the image becomes subordinate to operational thought. Prior to the advent of the operations, images are static. Anticipatory images frequently require conservation ability for which the preoperational child can only substitute pseudo-conservation. They may require transformations while he can only deal with states. With operational capability, children can be concerned with the transformations linking states. States can then be viewed as endpoints of some transformations and as starting points for others.

The basic distinction, then, between representational space and conceptual space is this: In the former, the image is basically static and attempts at conservation result in inadequate pseudo-conservation. Images govern thought. In the latter, the image can coordinate states and transformations. States are subordinate to transformations.

For Piaget knowledge is invariance through transformation. The mental image, insofar as it is static, is always only a symbol and is not in itself a form of knowledge. Operations suppose systems of transformations. Thus operations are more than images. However, images can serve as a tool of the operations.

Summary and conclusions. Visual space was differentiated from physical space. Physical space is external to the child's construction. Visual space is one component of sensori-motor space and later of perceptual space. In the early stages of perceptual space development, perceptual space and visual space may coincide. Later, perceptual space is a synthesis of what an observer "sees" and what he "knows." What he "knows" may emanate from representational or conceptual space. Therefore, when talking about perceptual space in adults, one must be careful not to attribute to perception what rightfully is representational or conceptual.

Researchers must be aware of the various aspects of the child's space. A geometry that models one aspect may not model another. Also, though the various spaces exist simultaneously in a child, they exist in various degrees of development. With the exception of visual space, which is more a function of the physiological characteristics of the child, each aspect of space is a function of cognitive development. Different aspects of space dominate the child's construction of space during different periods of his cognitive development. With the advent of operations, representations become mobile and serve as tools of the operations.

A Research Model

Some Assumptions

Piaget emphasizes invariability through transformations. He opposes the view that knowledge is a passive copy of reality. To know reality one must assimilate reality into a system of transformations, a system of transformations which attempts to model isomorphically the transformations of reality (Piaget, 1970). Knowledge is invariance through transformations. Klein also emphasized invariability through transformations. The notions of Piaget and Klein would seem to dovetail nicely into a model for studying the child's concept of space. While it is true that Piaget and Klein each use and even emphasize many of the same terms, to make a vague appeal to Klein's Erlanger Programm and to Piaget's epistemology as providing a model serves little purpose. In what sense do they or could they provide a model of the child's conception of space?

The position taken here is that the appeal to Klein and Piaget should be based on their common emphasis on groups of transformations and invariance through transformation. The assumptions made here are:

1. Piaget's definition of the nature of knowledge is essentially correct. Knowing requires construction of systems of transformations. These transformations become progressively more nearly isomorphic to transformations of reality. They eventually are combined into systems modeled by the mathematical group.

2. It is the technique of Klein's classification that is primary. It emphasizes a set X, a group of transformations, and invariance through transformation. This technique can be utilized to study the structure and sequence of the child's construction of his spatial reality.

3. The various classification hierarchies resulting from Klein's definition of a geometry are secondary. They can provide organization on the basis of transformations. They may be used to generate many research questions. However, it is premature to make any claim that any particular hierarchy models the sequence or the structure of the child's construction of his spatial reality.

Some Research Questions

Mathematics educators can raise many questions about the child's concept of space from a study of the Erlanger Programm. Knowing requires constructions of systems of transformations. What transformations do children use? Transformations can be studied in terms of their invariances. What are the invariants in the child's conception of space? Transformations must act on something. What is the set of points in which the child uses his transformations? It appears obvious that the answers to these questions depend on the child. Is there a sequence, common to all children, through which children naturally develop? Does this order parallel any hierarchy suggested by the Erlanger Programm?

Sequence. Piaget and Inhelder (1967) contend that the child's representational space is predominantly topological in nature until about six years of age. The child's first spatial concepts are those of proximity, separation, order, enclosure, and continuity. These are "topological relations" to Piaget. Deriving from these topological relations are the projective and Euclidean concepts.

Piaget does not always use mathematical language as precisely as mathematicians might desire. Consequently strong inferences from his work should be made only with caution. Piaget's topological tasks have been analyzed from a mathematical point of view (Martin, 1976) and the difficulties involved in making inferences pointed out. More analytical studies are needed which examine the mathematics involved in Piaget's other tasks e.g., those dealing with projective transformations (Piaget & Inhelder, 1967) or those dealing with mental imagery (Piaget & Inhelder, 1971).

On the basis of evidence now available, it appears that certain topological concepts such as interior, exterior, and boundary and primitive forms of proximity and separation develop early. Other concepts such as topological equivalence, order, and continuity evidently develop later. Probably some projective concepts also develop early, earlier than many topological concepts. More evidence is necessary.

"Inherent" sequence in mathematics may actually be only an organizational aid employed by mathematicians. That is, it may be merely a way to organize knowledge. Consequently, the Erlanger Programm does not impose an inherent sequence to the study of geometries, much less a sequence to the child's construction of space. What seems logically a prerequisite for attainment of a concept may appear so only as a result of a particular organization of the mathematics. If one takes the position that topology is the most primitive geometry because it contains the others, would one then say that the real number system is the most primitive because it contains the other number systems, e.g., the rationals, integers, and naturals? Structure does not automatically determine the sequence of the child's conceptual development. However, the Erlanger Programm does offer many alternative sequences to test as models of the child's sequence.

Continuous functions. Topological transformations are continuous transformations. They also have continuous inverses. Many continuous transformations do not have continuous inverses. It seems logical and consistent with Piagetian theory to expect notions of continuous functions in general to precede the notion of homeomorphisms, since inverses are not involved in the former. Proximities are the most elementary spatial relationships to Piaget. Proximities are preserved by continuous transformations, loosely speaking. A natural question is "What is the nature of the child's concept of continuous functions?"

Neighborhoods. If the child's spatial reality is essentially topological in nature before developing to incorporate projective and Euclidean concepts, homeomorphisms, the transformations of topology, have to exist without access to a metric. Homeomorphisms are continuous functions. Continuity was defined earlier using both metric and neighborhood notions. If length is not an invariant to the child, it would seem that his notion of continuity would have to be based on the neighborhood definition. Not much is known about the neighborhoods or proximities that a child uses in his space. Are his basic neighborhoods the same as, or can they be induced by some function from, the "usual" basic neighborhoods the mathematician uses in Euclidean space? The topology of the child could be drastically different from the usual topology of Euclidean space. Consequently, it could yield a reality quite different from a Euclidean reality.

Ideal points. It was shown earlier that projective transformations are not a subgroup of homeomorphisms in ordinary Euclidean space. Ideal points were introduced to Euclidean space. This is a mathematical convenience. Is it also a part of the child's construction process or are ideal points simply mathematical inventions? One's first reaction might be that it is silly to expect a child to have a concept of ideal points. Yet the Renaissance painters saw "points at infinity" in their visual space. And all of us are familiar with the "illusion" of the railroad tracks meeting in the distance. Ideal points could in fact exist in the child's space. They could be abandoned later because he cannot find a place for them in his developing Euclidean model of space. This would be analogous to our restricting real projective space to Euclidean space when moving from the projective transformations to the affine transformations in the discussion of the hierarchies of various geometries.

Recall that only one ideal point was associated with each line. Note that the railroad track illusion suggests the tracks meet "at both ends." Would the child consider these "intersections" as being at the same point or at different points? If the answer is two points, perhaps the child's space has elements of a non-Euclidean geometry.

The previous section discussed neighborhoods. If the child's space contains ideal points, what do neighborhoods around these points consist of? Intuitively one would expect a neighborhood of an ideal point to consist of points "far away" and in "about the same direction" as the ideal point. This could be described mathematically as follows. Consider a plane with a set of coordinate axes as in Figure 14. Each line through the origin is taken as representing its family of parallels. The line is identified by the angle $0^\circ < \theta < 180^\circ$ that it forms with the positive horizontal axis. A neighborhood of an ideal point P associated with a line ℓ could be the interior of a hyperbola whose vertices and foci are on line ℓ. The asymptotes for the hyperbola are the lines identified by the angles $\theta + \varepsilon$ and $\theta - \varepsilon$, where $\varepsilon > 0^\circ$. Note that points "far away" in space from the origin, or the subject, would be in the neighborhood if they were "in about the same direction." Thus intuition is satisfied. Also there are many neighborhoods for each ideal point and each neighborhood contains many ideal points. Whether or not this description models the child's construction, if indeed he even has a concept of ideal points, is unknown.

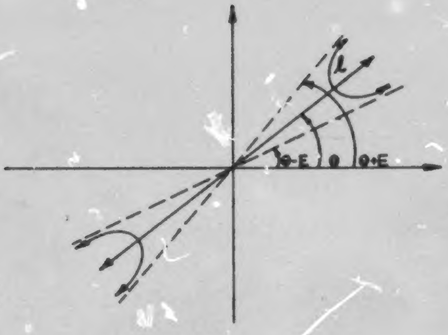

Figure 14. Neighborhood of an ideal point.

Ratios of distances. Similarity transformations multiply distances in all dimensions by the same amount. For example, the image of a triangle is a triangle whose sides are some constant k times as long as sides of the original triangle. The sides of the image are in proportion to the sides of the original. An affine transformation cannot multiply distances in all directions by the same amount (unless, of course, it is a similarity). However, affine transformations do preserve ratios of distances on the same line or parallel lines. In particular, they send equal distances into equal distances, thus preserving midpoints. The situation is illustrated in Figure 15. The rectangle ABCD is affinely equivalent to the parallelogram A'B'C'C'. Note the ratio AP/PB is equal to A'P'/P'B'. These are ratios along the same line. Note that the ratio AP/AR is not equal to A'P'/A'R'. These are ratios in different directions. A logical question from both the mathematical and psychological points of view is "Do children develop the ability to conserve ratios of distances in one direction prior to the ability to conserve ratios of distances in all directions?"

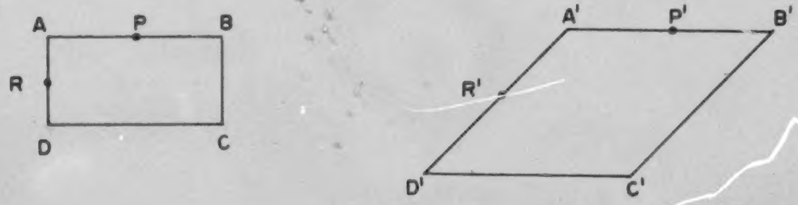

Figure 15. Ratios of distances preserved.

Constant of proportionality. Another question concerning proportionality can be raised. Similarities require conserving ratios in all directions. But they also allow the constant of proportionality to be any positive number. Euclidean transformations require conserving ratios in all directions also. But the constant of proportionality is always one. If one accepted a topological to Euclidean sequence of development, the Erlanger Programm would suggest that similarity transformations develop prior to Euclidean transformations. Yet, is it psychologically sound to expect a child to develop the concept of a variable constant of proportionality before the concept of a fixed constant of proportionality, whether or not the fixed constant is one?

Equiaffine transformations. As was shown in Figure 8, the equiaffine group lies intermediate to the affine group and the Euclidean group. Whereas similarities preserve shape but not necessarily area, equiaffinities preserve area but not necessarily shape. Does a child's space have any elements of equiaffine geometry apart from those inherent to Euclidean geometry? That is, can he preserve area without also preserving length? Have Piaget's area investigations used transformations which are not basically Euclidean to see if area conservation might precede Euclidean equivalence? There are vestiges of psuedo-conservation of area in the thinking of many adults. For example, many believe that a kidney-shaped region has the same area as a circular region provided they both have the same perimeter. In this case, conservation of one length produces psuedo-conservation of area. The interrelationships among similarities, equiaffinities, and isometries need study.

Isometries. Isometries have been the subject of more investigation than have other transformations and will not be discussed in detail here. However, one question can be raised. Do certain isometries develop before others? Mathematically, rotations and translations can be expressed as the resultants of reflections. Are reflections the most basic isometries psychologically? There is some evidence which suggests not (Lesh, 1975).

Group properties. The Klein Programm emphasizes groups of transformations. The group is not among the mental structures of the young child. Perhaps the child's conception of space develops first using a weaker structure. Semigroups require only that a set be subject to some rule of combination so that the resultant of the combination of any two elements is a unique element which is in the set and that the associative law holds. Semigroups do not require identity elements or inverses. It is possible that the semigroup would serve as a model for the child's mental structure. Using this structure a child could develop certain invariants of a geometry but not others. For example, the set of continuous transformations forms a semigroup. Proximities are preserved by continuous transformations whether or not these transformations have inverses. But a continuous transformation may not preserve openness of curves. Thus, a child whose mental structures were isomorphic to a semigroup would develop some invariants of a geometry. The development of others might require the group structure. Using the semigroup properties of the projective transformations, what invariants could a child construct?

Of course, transformations can be more general than continuous transformations, and structures can be more general than the semigroup. What about transformations that are just one-to-one, many-to-one, or the composition of transformations in general? Classification can be modeled by the many-to-one transformation. Perhaps other types of transformations could be used to model other structures. Much work could be done in this area.

The child's construction process. The Erlanger Programm can be used to provide geometric models of the spaces which a child constructs. It can focus research on transformations and invariances rather than on states. It can offer hypothesized sequences of the order the child's construction will follow. It cannot provide a model of the process of the construction.

To Piaget the essential aspect of thought is operative not figurative. The operative aspect deals with transformations. Knowledge is invariance through transformation. Conceptual space, as discussed in this paper, relies on operative thought. It follows that the construction process for conceptual space should be of primary concern. Investigations regarding perceptual space would derive importance only insofar as they deal with the foundations of the construction of conceptual space. Studies of the construction process for representational space would have comparable value particularly if a careful analysis of the beginning points or foundation for the construction process is provided.

In this paper, the structure of geometry via Klein's Erlanger Programm has been examined, and the various aspects of the child's construction of space (à la Piaget) have been discussed. From the union of Klein's Programm and Piaget's theory, many research questions have been generated. The Erlanger Programm offers a model which focuses research on transformations and invariances. However, it must be remembered that this model is offered only to assist the description of the product of the child's construction of space and not the process itself.

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A Dynamical Model for Cognitive Development with Applications to Problem Solving

Donald G. Saari

Northwestern University

Can problem solving be taught? I contend that, within limits, it can be. It is the purpose of this paper to provide a foundation for this claim. However, in order to discuss this issue (indeed, even before a definition of a problem can be put forth), we must develop a model for cognitive development. Once this model is derived, applications to problem solving, education, etc., will be immediate.

The proposed model is based on elementary first principles from the study of cognitive development. These principles will be in the form of basic minimal assumptions concerning the behavior of assimilation and accommodation in the adaptation-organization model used by J. Piaget. We shall assume these basic principles as the building blocks for the model and use some recent results from mathematics, namely, R. Thom's catastrophe theory, to show the relationships between these principles. It will turn out that while the inputs are elementary, the conclusions are sharp and far reaching. One of the surprising facts for me was that this dynamical model based upon the theory of differential equations provides an understanding as to why "attitudes" play such an important role both in the educational process and in problem solving. This was entirely unexpected, but it was most happily received. A second unexpected bonus was an analytical description of the well know eyewitness phenomenon.

For technical reasons which will be explained in the next section, the exposition of the model for cognitive development will be simplified by the simple device of restricting attention to a small, fixed portion of the environment. It seems natural to label such a restriction a problem, and we shall do so. Consequently, problems and problem solving are the main topics of discussion, where problem solving is viewed as a special case of cognitive development. However, in the context used here, the discussion of problem solving must be viewed as a description of the general ideas of cognitive development given by this model.

Once the model is described, some applications will be given. These sections are not intended to be complete or comprehensive. Instead, applications will be restricted to those conclusions which follow in a natural fashion from this model.

There are two reasons these sections are included. The first is to further illustrate and explain the model. The second is to aid researchers interested either in applying or in experimentally verifying this predictive model. Indeed, this is a theoretical model, and for the most part, it requires empirical supporting evidence before its ramifications and limitations can be completely understood.

This paper is an encapsulation of some of my conclusions resulting from a study of cognitive processes (Saari, in press-a, in press-b). All of the points raised here will be elaborated upon in greater detail in this reference, which can be viewed in a more general context of being a study of the adaptation process. Indeed, even in the abbreviated version provided here, the reader should find little difficulty in applying some of the conclusions to other models--say from psychology or economics--which depend upon an adaptation process.

I would like to thank R. Lesh and B. Chartoff for their patience in listening to these ideas while they were still in a formative crude stage. I would like to give particular thanks to Edward N. Saari for the several informative discussions we had exploring the topics of creative thinking in the visual arts and general problem solving. Several of his points turned out to be most helpful and useful, particularly some of his observations concerning the discontinuities inherent in problem solving. Some of the examples used here are due to him.

Cognitive Development

We shall start off with some elementary first principles of cognitive development. These statements will be rephrased in mathematical terminology, where we will need to appeal to some recent mathematical results to complete this translation into a mathematical language. This translation will constitute our model for cognitive development.

Start off with environment E where we shall ignore any given structure. This is a key point since our underlying assumption is that any environmental structure we perceive is imposed upon the environment by us--an interpreting organism. But how does the organism do this? Assume a message, or event, $e \in E$. Message is "interpreted" by a matching of this message to some existing organizational structure of the organism. The organizational structure to which this message has been assigned provides the organism's interpretation of the event.

We shall be vague about the form of the organizational structures since this is of little interest to us here. They could be interpreted as elements of the long-term memory, as schemata, etc., but an exact definition or choice of terminology will divert us from the goal of finding a description of how the organizational structures change.

Whatever these structures may be, there must be only a finite number of them. The physical limitations of an organism dictate this. Thus, we shall assume there exist N possible organizational structures, where N is a (possibly very large) finite number. All possible organizational structures are the closed positive orthant of an N dimensional Euclidean space R_+^N. A given organizational structure is represented by a vector $\underline{X}$ from this space, where the kth component of this vector represents the kth organizational structure. $X_k = 0$ implies the kth structure is inoperative, while a large value for X_k implies that the kth structure can handle, or permit the interpretation of a large number of messages from the environment. At any given instant of time, t, not all of these potential structures are developed. This is a direct consequence of the fact that the value of N must be so large that all "reasonably" possible organizational structures are represented as a component of vector $\underline{X}$. Consequently, at any time the values of most of the components will be either zero or close to zero.

So far we have represented the environment and the space of possible organizational structures respectively by E and R_+^N. We now need to represent the process which transmits the message e from E into R_+^N. This procedure of attempting to match a message from the environment to existing organizational structures is called the assimilation process, which we shall denote by As. This process consists of two potentially conflicting parts. The first is discriminatory assimilation which is the process of examining a message to discover its particular characteristics, to determine how it differs from other messages, etc. The second is generalizing assimilation which is the process of searching for agreement between this message and others, of finding a general class to which this message belongs. We shall denote the former by As_d and the latter by As_g. Clearly these processes change in effectiveness with time, due to changes in motivation, etc. In this communication we shall not address the important problems of how to establish discrimination or generalization behavior in a learner. Rather, we shall concentrate on fitting these capabilities within a model for cognitive behavior of large scope.

All of this can be expressed mathematically as

$$As_d,\ As_g\colon E \times R_+ \longrightarrow R_+^N\ . \tag{1}$$

That is, the two types of assimilation depend upon time, at least indirectly, and they match messages from the environment to organizational structures.

If the image of $\mathrm{As}_g(e,t)$ agrees completely with that of $\mathrm{As}_d(e,t)$, then there are no difficulties, and the message is interpreted as the common image. However, we cannot expect this to always occur, even for the same message at a different time. Therefore, in general we can expect that the images of these two processes do not match precisely, that is, they are in at least partial conflict. This state of conflict in the interpretation of a message is our definition of a nontrivial problem.

To resolve this conflict, something in the system has to change. Changes in the organizational structure in response to a nontrivial problem is called the accommodation procedure. Namely, it is the process whereby the organizational structures must be modified to fit the message from the environment. This is the process one must understand if there is to be any hope in modeling cognitive development.

Examine the above description. Before the accommodation procedure can change the organizational structures, there must be an initial interpretation of a message. Consequently, it follows that the accommodation process, denoted by Ac, depends upon $\mathrm{As}_d(e,t)$ and $\mathrm{As}_g(e,t)$. Indeed, accommodation is the process whereby the system attempts to find a new structure which will eliminate the conflict created by these interpretations. Notice that accommodation is a part of the cognitive system; we are not assuming it is a conscious overt act of the organism. Thus, in some crude sense, Ac determines the amount of change in the structures which is needed to eliminate this conflict, or

$$\mathrm{Ac}[\mathrm{As}_d(e,t), \mathrm{As}_g(e,t), -]: R^N_+ \longrightarrow R^N \qquad (2)$$

where the value of the kth component of the image corresponds to the amount of change required in the kth structure to reach equilibrium or a state of no conflict. Since the kth structure may need to decrease, the image space is R^N, not R^N_+. Perfect agreement in the message corresponds to an image where none of the structures need to change, that is, to the image $\underline{0} \in R^N$.

Before continuing, we would like to reduce the number of parameters to a manageable level. The problem is that As_d and As_g depend upon all possible $e \in E$. Consequently, we are potentially dealing with an infinite dimensional parameter space. Therefore, restrict attention to a given message e or to some small neighborhood of e in E, for example, some subject field. The idea of a small neighborhood makes sense since some sort of structure or topology has been imposed upon E via the inverse images of the assimilatory processes. Namely, we can assume E has the weakest topology making these maps continuous. Furthermore, we can safely assume that the initial interpretation, or assignment, or $e \in E$ depends upon the respective talents, degree of sophistication, and motivation of $As_g(-,t)$ and $As_d(-,t)$. Assume there is some sort of metric or norm which measures this drive or motivation; $\|As_d(-,t)\| = d \geq 0$ and $\|As_g(-,t)\| = g \geq 0$. Zero values mean the corresponding processes are not working. Larger numerical values for these parameters correspond to a more talented, able, and motivated process. Finally, let $\underline{X}$ denote the common portion of the images of As_d and As_g. Namely, $\underline{X}$ corresponds to that agreed upon component of the differing interpretations of As_d and As_g. (This dual usage of $\underline{X}$ should not lead to any confusion.)

These assumptions permit Ac to be viewed as a function from $R_+^2 \times R_+^N$ into R^N, namely the function which gives the vector from $\underline{X}$ to an equilibrium position corresponding to the given values of d and g. We are interested in the state of perfect equilibrium,

$$Ac(d,g,\underline{X}) = \underline{0}. \tag{3}$$

If Ac is a smooth function (in the sense of differentiability), then this equilibrium state is generically a smooth two dimensional manifold (a smooth surface which locally appears to be a two dimensional space). The interpretation of this equation is straight forward. For a given value of d and g, a value of $\underline{X}$ satisfying Equation 3 corresponds to a structure where the message is interpreted without any conflict.

There are three complications:

1. The process of accommodation is not an instantaneous one. The process does not arrive at the equilibrium position of Equation 3 immediately, but only after a period of adjustment and change.

2. The process of accommodation must admit discontinuities. It is a common experience and observation that a sudden change may occur in the interpretation of a message from the environment.

3. Not too much is known about the accommodation process. Therefore, any description of this process must be generic or stable in the sense that small variations reflecting changes in our understanding of the process or changes due to individualistic differences still should result in a similar qualitative description with the same conclusions. We shall consider these constraints in the order 1, 3, 2.

Constraint (1) claims there must be a period and process of adjustment between the assimilation and accommodation process--between the values of d and g, and the corresponding choice of $\underline{X}$. This process of reaching some sort of accord between As and Ac is known as the adaptation process. Since it is an adjustment process with the implied notion of rate of change, it is natural to model it with a differential equation, which for technical reasons, we assume is given by a gradient of a smooth function. That is,

$$\dot{\underline{X}} = \eta f(d,g,\underline{X}) = \eta\nabla F = \eta\left(\frac{\partial F}{\partial X_1},\ldots,\frac{\partial F}{\partial X_N}\right) \tag{4}$$

where F: $R^2_+ \times R^N_+ \longrightarrow R^1$ and where η is a positive parameter. A stable equilibrium position, that is, an element of a subset of the zero set of $f(d,g,\underline{X}) = 0$, is the equilibrium position of the Ac process. (For the remainder of this paper we shall, incorrectly, identify Ac with the equilibrium position given by Equation 1. Namely, we assume that the zero set of f has an attracting surface for the equation in some bounded region of R^N_+. For fixed d and g, the solution of Equation 2, $\overline{X}$ (t) tends toward some element of this zero set with coordinates $(d,g,\overline{\underline{X}})$. Recall, our space is $R^2_+ \times R^N_+$. The process of moving $\underline{X}(t)$ toward the zero set, modeled by the differential equation, is the adaptation process. The limiting position in the zero set of f is what we shall call the accommodation process.

It remains to choose function F satisfying conditions (2) and (3). It turns out that this seemingly impossible task can be accomplished by using our basic assumption that parameter d and g are basically in conflict; that is, there does not exist a smooth combination of these parameters which would allow Ac or F to be viewed as a function of a single parameter. For expository reasons, we shall discuss our choice of F in the special, unrealistic setting N = 1, and then describe the model for arbitrary finite N. That is, we shall initially discuss the model for an organism which possesses only one organizational structure, and then generalize it to a more realistic setting.

Recent developments in mathematics, known as catastrophe theory and primarily due to René Thom (1969, 1972), require that the local behavior of the zero set of a function f exhibiting constraint (2) and satisfying constraint (3) must be as given in Figure 1. Recall, N = 1. The probable global behavior of the zero set of f can be pieced together from this knowledge of the local behavior and the Ac process. Also, we need to establish the orientation of the cusp fold in the space $R^2_+ \times R_+$. To determine this, consider the behavior of Ac with respect to changes in the values of d and g.

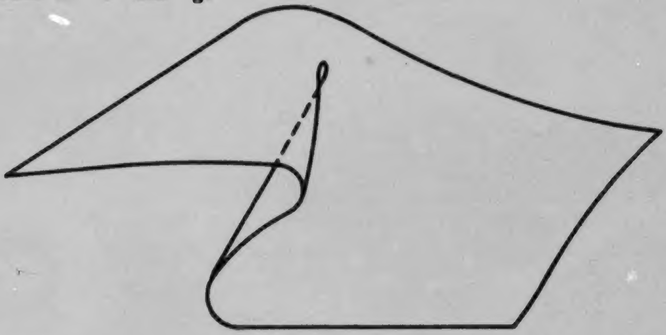

Figure 1. Representation of surface for f = 0, N = 1.

Fix d at a small value and let g increase. This means that the process of assimilation is generalizing, noting more similarities between messages from *E*. Consequently, after the adaptation process has been completed, the organizational structure can incorporate a large number of messages from *E*; this means that small fixed d but increasing g leads to increasing values for $\overline{X}$. This does not mean that the structure is "better," just that it is interpreting a larger subset of *E*. It may be doing so "unwisely."

To determine the impact of parameter d, fix g at a fixed value and let d increase. This means the differences and the discrepancies in messages are noted and emphasized. This causes problems in trying to interpret messages within the one existing structure, which results in a smaller subset of *E* which the structure can interpret with any confidence. Thus, small fixed g and increasing d leads to a decreasing $\overline{X}$.

In the above two cases one parameter dominated the other. Turn now to the case where one parameter starts from a position of strength. Let g be large and d be small. As we have seen, this corresponds to a large value of $\overline{X}$. Hold g fixed and let d increase. This means that there is an increase in emphasis on the differences between messages. However, the noted similarities between the messages are so strong (large g) that this increase in d has little effect on the value of $\overline{X}$. As d continues to increase in value, more discrepancies in the messages are noted, leading to a decrease in the value of $\overline{X}$. Nevertheless, the amassed evidence is still not sufficient to question seriously the accuracy of the interpretation. This process can be expected to continue until some critical point where the accumulated conflict and the weight of the discrepancies of the message so undermine the interpretations of the one structure organism that there is a sudden decline in the value of $\overline{X}$ corresponding to a sudden reduction to a "safe" level of interpretation. (Of course, all of this is in the case $N = 1$. For larger values of N, this "discontinuity" in $\overline{X}$ may be manifested by the division of one structure into several other structures. At the same time, different structures may be combined into a new one.) Thus, we can expect that if a structure will experience a discontinuity, then it occurs for large values of d and g. Indeed, a larger initial value for g, implying a stronger, more stable generalized structure, would require a larger value for d before the structure breaks down.

This leads to the general positioning of the cusp fold in Figure 2. (We shall interpret this figure in the next paragraph.) The arrows in the figure correspond to Equation 4, and the action of the differential equations. That is, they denote the adaptation processes of changing the structure and forcing $\overline{X}(t)$ toward the equilibrium surface which denotes perfect agreement in messages for that level of d and g (plus the restriction on *E*). Notice that not all of the zero set of f corresponds to Ac. The "tuck" in the surface between the upper and lower surface is the location of the unstable points (the arrows tend away from this part of the surface).

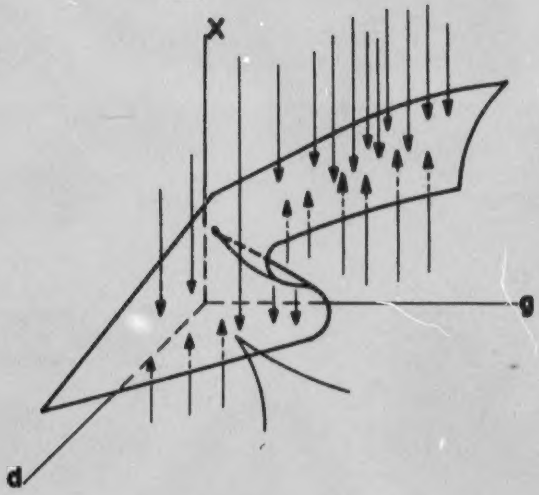

Figure 2. Probable orientation of the adaptation equilibrium surface in d-g-X space, $N = 1$. The vectors represent Equation 4.

The best way to interpret all this is to consider the consequences of changes in the values of d and g. A path moving in the d-g plane, will have its image forced toward the surface by the adaptation process, the vector field in Figure 2. For example, consider the two different paths in Figure 3. (To determine the images of these paths, compare Figure 3 with Figure 2. The cusps in Figure 3 correspond to the location of the boundaries of the folds in Figure 2. That is, they are the image of a vertical projection of these folds into the d-g space.) The images of the paths from 1-2 and 2-3 can be only on the upper surface. The path from 3 to 4 crosses a branch of the cusp fold. However, this branch corresponds to the fold on the lower surface, so it has no impact upon this path, and the image stays on the upper surface. Notice that path 4-5 crosses the second branch of the cusp, and this branch does correspond to the fold on the upper surface. Therefore, once the path crosses this branch, the adaptation vector field forces the image to the lower surface. Thus, we have the apparent discontinuity in the structure discussed above. (In fact, it is not a discontinuity, but rather a relatively rapid change in the value of $\underline{X}$.) The analysis of the second path is similar, except notice that the image of point 4 for the second path differs from that of point 4 for the first path. Notice that the changes in the values of $\underline{X}$ are continuous except when a path crosses the second branch of a cusp. Thus, the cusps in the d-g plane provide valuable information in that they mark the locations of the discontinuities of the process.

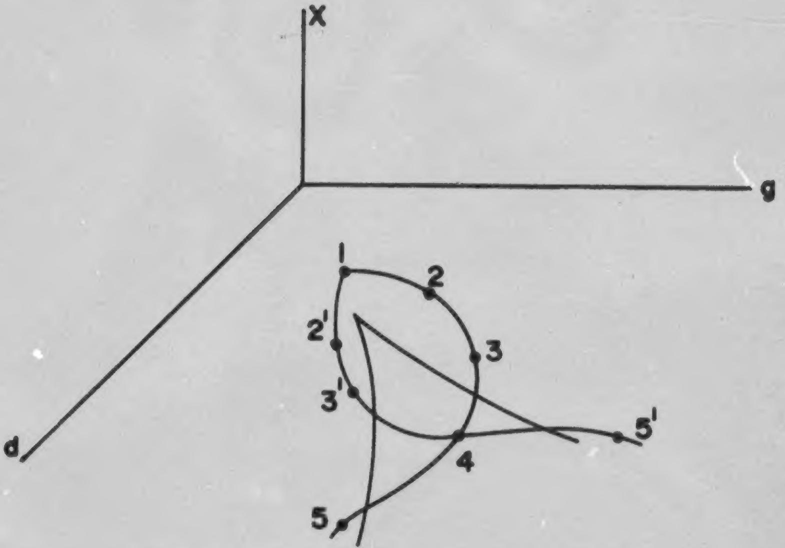

Figure 3. Two paths in d-g space, N = 1. Notice that the cusp is a projection of the boundary of the surface in Figure 2.

It turns out that Thom's theory still holds for arbitrary finite N. Locally, the location of the discontinuities emanates from cusps. Arguments similar to the one used for N = 1 show that the general positioning of these cusps is outward and centered about a diagonal line somewhat in the center of R_+^2. The main difference is that a discontinuity may result in some rapid changes in several coordinates, reflecting the general bifurcation of organizational structures. The main idea to remember is that it is the "second" branch of a cusp which a path crosses before there is an apparent discontinuity. Also, in the more general setting of arbitrary N, the cusp pertaining to a given path depends upon which surface is the image of the path. (In Figure 4, we have given an example of a parameter space for N > 1.)

Two more points need to be explained before the model is completed. The first is positive parameter η. This parameter determines the operating rate of the adaptation procedure: the larger the value of η the faster the process. Indeed, $\eta = \infty$ corresponds to an infinitely fast process. Parameter η is included to allow for individual differences in the rate of adaptation, perhaps of a physiological nature.

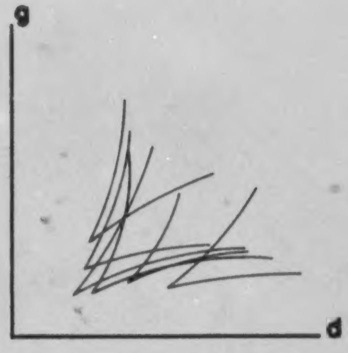

Figure 4. Schematic representation for $N > 1$ of the projection of the boundaries of the equilibrium surface onto the d-g parameter space.

It follows from the above discussion that the process of change in the organizational structures depends upon the selected path in the d-g space. However, movement of a path presupposes some interpretation of the preceding events. Namely, the rate of change of a path in the d-g space is related to the speed of the adaptation process: an equilibrium position must be approached, meaning that new structures are formed or old ones extended before the path can continue. We repeat this fact as a hypothesis:

Hypothesis. The rate of change of a path in the d-g space must be "slow" compared to that of Equation 2, that is, compared to the value of η[1].

The fact that the rate of change of the assimilation process (the path in the d-g space) depends upon how fast some sort of accord can be reached with the accommodation process completes a circle illustrating the strong interrelationship between the assimilation and accommodation processes. The extended structures and the new interpretations permitted by the accommodation process are necessary before a change in the values of d and g can occur. However, this relationship, as modeled here, is a continuous dynamical one. While the two processes operate simultaneously, there is a strong feedback mechanism between them which acts as a check on their movement. It is the conflict in the interpretation of the messages which is the driving force for the path in the d-g space. Movement of the path changes the structures, but these changes in the structures must be realized before the path can continue.

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[1] For a given choice of F, a mathematical description of "slow" and its relationship with η can be found in Levinson (1950).

Consequences

We can now reach some conclusions about cognitive processes. Some of the more obvious ones will be given here, and a more complete list will appear in another study (Saari, in press-b). However, the reader should find it easy to compile a fairly complete list, since the procedure is to examine the consequences of different paths in Figures 3 and 4.

The first observation is that the accommodation process and hence the subsequent organizational structures depend not only on the capability and drive of the assimilation processes (i.e., the values of d and g) but also upon the path taken to reach these values. This means that the learner's current organizational structures depend very strongly on the learner's history. This is reflected in Figure 3 by the fact that the image of point 4 can only be determined if we know its history, namely, to which path it belongs. In Figure 4, with its wealth of cusps, we see that two paths arriving at the same point in the d-g space, but with different histories, may result in images where completely different components in $\underline{X}$ are emphasized. This may result in radically different interpretations of the same message from *E*. Since this is what we expect, it adds partial confirmation to this theoretical model.

What can this mean for education? Consider an example of teaching elementary arithmetic. The goal is to teach both comprehension and ability to use basic addition, subtraction, and multiplication facts. Suppose that in learning these facts, most of the students in a class took a route in Figure 4 similar to that of Path 1, 2, 3, 4 in Figure 3, and they have attained a certain level of comprehension and competency in arithmetic. Now suppose one member of the class learned these facts by rote. That is, essentially, the child viewed each new fact as a separate entity. It is the differences in the facts which are impressed upon him, rather than any similarity between them. Thus, the g parameter remains at a low value while the d parameter is increasing. This corresponds to a path similar to 1, 2', or 3' in Figure 3.

The problem is to help the child reach class level. A natural approach may be to try to direct his path to point 4. Although, when the child has reached this point, his assimilation talents are equivalent to class level, his understanding is not; he is still on a different surface. It follows from Figures 2 and 3 that the path needs to move to the vicinity of point 5' before a new level of comprehension is achieved.

It is questionable whether this is a feasible path. For example, the increase in the g parameter required to cross the second branch of the cusp may require generalizing examples or experiences beyond the capacity of the teacher to provide them. Second, it is not clear this path is optimal in terms of time and the required effort. A more efficient path may be one which crosses the second branch of the cusp near the vertex. This requires a decrease in d or a return to the beginning first principles.

While this example is elementary, it does illustrate how this model can be used to derive conjectures or explanations for educational problems. For example, notice that this example shows the importance of recognizing and correcting a problem of this type at an early stage when the distances between branches of the cusps are still small. This leads to our second observation.

Larger values of d and/or g tend to lead to more stable structures. That is, small changes in the values of d and g are not as likely to reach the appropriate branch of the cusp causing a discontinuity in the organizational structures. Examples supporting this statement are abundant. We can see this in the stabilization of certain parts of our beliefs with time. It indicates why people new to a field are somewhat more receptive to innovative ideas; it points out the impact and importance of impressions coloring early interpretations of events in the environment.

However, since we can expect the distance between the surfaces in the fold to increase with the distance from the vertex, if a change occurs for large values of d and g, it may be a dramatic and significant one. Consequently, while the above paragraph suggests the impact of indoctrination at an early stage, this observation suggests a possible later consequence--a large discontinuity which may manifest itself as a complete rejection of the original interpretation. We leave other examples to the reader.

The third observation is a somewhat surprising one. While the accommodation process is path dependent, it does not depend continuously upon the choice of paths! Namely, two individuals with almost identical histories and abilities can reach different interpretations of a message. To see the origin of this statement, trace the image of two nearby paths which pass on opposite sides of a vertex of a cusp. The image of one path will be on a different surface. The implications this statement has for education are obvious. For example, it explains why a class can be at two different levels of comprehension. This discontinuity in Ac will be exploited in our discussion of strategies for problem solving.

Problems and Problem Solving

Problems are usually defined as an impediment, restriction, or delay in the stimulus-response mechanism. A problem, then, becomes a stimulus for which a response is not immediately forthcoming: a question for which there is no immediate or obvious answer.

In our setting we have defined a nontrivial problem to be a conflict in the images of $As_d(e,t)$ and $As_g(e,t)$, a definition that agrees with the above. However, for the model such a definition is incomplete (we shall explain this at the end of this section), and we need a more general definition.

Definition. A problem is a message e from the environment which is assigned to a nonzero organizational structure. If $As_d(e,t)$ agrees with $As_g(e,t)$, then it is a trivial problem. (Assignment of the message to a zero vector means the message is not being interpreted.)

A method of problem solving is a path in the d-g space. A successful method is one for which agreement in the images can be attained in finite time. We assume these paths, $\gamma(t)$, are continuously differentiable. Thus, a successful method is one for which $\gamma'(t_1) = 0$ for $t = t_1$ in some compact interval. For a trivial problem, the path is a stationary one.

No value judgement is implied by the term "successful." For example, a successful method may be one for which the terminal value of some parameter (d or g) is less than its initial value. This may correspond to a rejection of the conflict by ignoring the implications of the original interpretation. On the other hand, this path may correspond to a return to first principles. Nevertheless, it is reasonable to label such a path as a regressive method of problem solving.

The stages of problem solving can now be read directly from the model. It is important to understand these stages because they form the foundation for problem-solving strategies. Of course, once we have problem-solving strategies, the major obstacle to the teaching of problem solving has been removed. It will turn out that several of these stages can operate simultaneously. Thus, a first order approach to problem solving is to assist the process by concentrating on each stage separately and in the proper order. Since these steps for problem solving will parallel the stages of problem solving, they will be discussed together. The step corresponding to a given stage will be designated by a primed number.

1. Transmittal of the message. There is a message from the environment. It is transmitted by the sensory system, and it must be assigned by As to some existing organizational structure.

1'. Pay attention. Understand the message. The resulting problem depends upon what percentage of the message is received. Therefore, pay attention to the message.

Even a casual examination of the model shows that the initial interpretation of a message is most crucial. Consequently, the assignment of a message to a structure must be done rationally. The message should be carefully examined and understood before an initial interpretation is accepted.

2. Conflict in the interpretation. If the message is a trivial problem, the process is completed. If there is conflict in the interpretation of the message, then the nontrivial problem must be solved.

2'. Understand the problem. The problem must be studied to determine why it is a nontrivial problem. The conflict, weakness in the interpretation, and what additional information is needed should be established. This is the first place where attitude plays a role. Students should realize that conflict and a lack of complete understanding or the lack of an initial plan of attack are most natural.

3. Preliminary movement of the path. It is the conflict caused by a nontrivial problem which is the feedback mechanism forcing changes in the values of d and g. This change or movement of the path is an attempt to solve the problem by reinterpreting the message. It is aided by gathering of information (previously interpreted messages), etc.

3'. Accumulation of information. In step 2', the problem was defined and the weak points isolated. In this step the conflict and weak points are studied and additional information is sought. That is, the problem, or at least different aspects of it, are compared with other events, usually already interpreted, to determine differences and similarities. This new information changes and/or extends the existing organizational structure. It may result in a plan of attack. (This is a most important step. Some additional techniques suggested by this model will be discussed in the next section.)

4. Adaptation disequilibrium, or incubation. The path in the d-g space may be moving "fairly fast." This would move the image of the path away from a region near the equilibrium surface. According to our hypothesis, this leads to a period of disequilibrium. During this period, movement of the path in the d-g plane is slowed until the adaptation vector field can force the image, $\underline{X}(t)$, back to a region near this surface. When $\underline{X}(t)$ is away from a region about this surface, it is changing rapidly, which would lead to a state of confusion or no progress. The interpretation of e is not clear since the distance of $\underline{X}$ from the surface measures the conflict in these interpretations.

A second and more dramatic type of adaptation disequilibrium occurs if the path crosses the appropriate branch of a cusp leading to a discontinuity. This period may be longer. During both types of incubation periods, there may be no external signs of work or activity.

4'. *Positive attitude, incubation*. Just by its nature, the state of disequilibrium can be a confusing and frustrating one. During this period, take a short vacation from the problem--take a walk or read something else, but give the adaptation process the time it needs to do its work. This is easy advice to accept because it follows *3'* which is a period of hard work. However, by a recess, I do not mean the problem should be abandoned. A positive attitude toward the problem is needed. This approach is needed to counter possible negative side-effects of this state of disequilibrium, a period which *may* be marked by irritability, need for privacy, doubt, etc. (depending, of course, on personality traits). Without a positive approach, a natural mode of problem solving might be the regressive method. Namely, a path is selected which eliminates the conflict by ignoring part of it.

This is somewhat ironic since the greatest degree of conflict should appear near the branch of the cusp leading to a discontinuity or an incubation period of the second type; after all, it is the conflict near the branch which leads to the discarding of the old structures and an attempt to assign new ones. However, this is also the period where, without a positive approach to the problem, a regressive method will most likely be adopted; consequently, *the threshold line may not be crossed*.

5. *Equilibrium and inspiration*. When $\underline{\overline{X}}$ does approach a region near the equilibrium surface, that is, at the conclusion of an incubation period, there is a new interpretation of the message. Since the adaptation vector field is "fast" compared to changes in the path in the d-g space, this new interpretation may appear quite suddenly--an inspiration. After an incubation period of the first type, the inspiration is typified by a "Oh, that's right; of course." type comment. After an incubation period of the second type, the equilibrium position corresponds to a reassignment of formation of new structures to interpret the message. Thus, this inspiration can be characterized by the exclamation "Oh my gosh! So that's what happens!"

5'. *New plan of attack*. The inspirational period provides a different interpretation of the problem. This should be analyzed to see if it provides a new plan of attack for the solution of the problem.

6. Confirmation. The message is examined in terms of the new structures to determine whether this interpretation solves the problem and resolves the conflict. If not, the process is continued. Notice, if this reassignment of the structures is of the second type and if the projection of this surface onto the d-g space, R_{+}^{2}, indicates that our path is near the vertex of a cusp or a threshold branch (with respect to the path) of the cusp, then we might expect another reassignment shortly thereafter. However, if the path is "far" from the threshold branch of the appropriate cusp, then we can expect that this was a stable reassignment.

6'. Carry out the plan of attack. Return to step 2' and determine whether the problem is solved. Continue on this loop of steps until a solution is obtained either by a regressive method, which includes giving up, or by finding a solution.

7. Completion. The problem is solved. If the problem was not a trivial one, then the concluding structures differ from the initial one. The values of some of the components of $\underline{X}$ have changed. The concluding structures will be used for the interpretation of new similar messages from the environment. Indeed, it may even provide a new definition for a neighborhood of the message in E. This is discussed in greater detail in Saari (in press-b).

7'. Examine the solution. Since the resulting structures will be used for the interpretation of future similar messages, the structures should be strengthened. Namely, the solution of the problem should be examined until it is completely understood. (What were the techniques used? Why did they work?)

This completes an outline of the stages and steps of problem solving. Notice that these stages hold even for the regressive method of problem solving; however, the steps are intended to avoid this approach.

We conclude this section by briefly explaining why "trivial problems" were included in our definition of a problem. The main reason is a mathematical one. A method of solution is a smooth curve, but we did not specify a minimum length for the curve. Therefore, for reasons of completeness, problems with solution curves of zero length should be included, but this is our definition of a "trivial problem." Thus, our definition of problems includes the 2 + 2 variety of simple problems. This is assuming, of course, that the learner is sufficiently advanced so that 2 + 2 is trivial.

A trivial problem is trivial because its solution path has zero length or approximately zero length. This does not mean it is elementary in the usual sense of the word. We will give an example, using the above limit argument on the length of a path and the incubation period to illustrate this point.

Consider an unfamiliar event in the environment which is short-lived but replete with details, such as a short advanced lecture. The first stages of problem solving are to receive the message and assign it to some existing organizational structure. However, the lack of familiarity and the number of details lead to conflict in the interpretation and assignment of the message. This provides the impetus to move the d-g parameters. The number of details and the short duration of the event leads to an attempt to move this path very quickly. According to our hypothesis on the speed of the path, and our model, this will tend to move the image of the path away from the equilibrium surface. Consequently, the path cannot move substantially forward until the image is near the equilibrium surface, that is, until the current data or message is understood. However, by the time this incubation period has ended, the details are past this point and are confusing. This leaves the regressive approach as the only viable method of problem solving--we turn him off. Thus, our terminal interpretation is nearly the same as our initial, and perhaps incorrect, assignment of the message. Notice, this suggests that if the lecturer is to "keep" a large portion of his audience, he should help in the assignment of the message. This should be done by assigning it to a stable structure, which is presumably an elementary concept.

Now, decrease the period of time allowed for the message, in this case, the lecture. The above shows that the difference between our initial and final interpretations of the message is small, and in the limit the two interpretations are essentially the same.

At this point our example of a lecture breaks down. Therefore, consider a sudden, short-lived, and unexpected disruption in the classroom, or in some public facility. In this case, the time period of the message closely approximates the limiting process described above, and the initial and final interpretations of the event are essentially the same. Therefore, this must be classified as a trivial problem. The accuracy of the interpretation of the event, as compared with the interpretation arrived upon if the event could be replayed in slow motion, would depend upon the accuracy of the initial assignment. (How unexpected or unfamiliar was the event? How experienced is the observer and what are his prejudices, i.e., what are his existing organizational structures?) No matter how "inaccurate" the interpretation may be, the observer has no reason to doubt it. This is, of course, the well-known eyewitness phenomenon. Thus, a trivial problem may be anything but trivial.

Strategies for Problem Solving

In the preceding section the steps of problem solving were oultined, but it is not entirely clear how to implement some of them. Therefore, in this section we shall briefly discuss some strategies for problem solving suggested by this model.

1. Optimal choice of paths. While we have not introduced the concept, the scratch pad for problem solving is our Short Term Memory (STM). STM holds at any time a maximum of somewhere between five to nine symbols, and this upper bound decreases when an unrelated task is being performed. Thus, for any task involving the combination of a large number of concepts or computations, STM will require aid. This can be done through external aids such as tape recorders, computers, paper and pencil, etc., but it also can be accomplished by combining several ideas or concepts into a single class. Carried one step further, this suggests that in order to avoid overloading STM, an optimal path is one which emphasizes the similarities between the messages first, the g parameter, and then it considers the differences between them, the d parameter. (It may seem to the reader that we are cheating since STM is not part of our earlier discussion. However, in a more general discussion (Saari, in press-b), STM becomes an integral part of the model.)

In addition, it is suggested from the model that a path of the type described above results in structures capable of interpreting a larger portion of the environment, which is clearly a desired state. Also, this type of path seems to be the route which leads to intuition. The intuition is gained by a generalization--the lumping of a given message with a large class of other messages. The details are checked later.

With this knowledge of the optimal path, any technique which aids the development of the g parameter will be a most useful technique for problem solving. For example, this includes collaboration, discussions with others, presenting a lecture on the problem (that is, discussing the problem without external aids for STM, which forces a more general organization of the material), studying similar problems in other regions of the environment, etc. Most of the statements which follow in this section can be viewed as additional methods to realize this optimal path.

Notice, however, this choice of a path suggests that a method which emphasizes complete understanding of each aspect of a problem before proceeding to the next one may not be optimal! Furthermore, it suggests that in research, the approach of shooting to the boundary of research and then picking up details at a later date may be a positive approach.

2. Deferred judgement. The principle of deferred judgement should be employed. Namely, ideas, plans of attack, and options should be explored before they are evaluated and discarded. This is, of course, the basic

premise behind brainstorming. But I mean it to apply to all types of problem solving, for example: a photographer deeply involved in his work should defer final evaluation of a given print until a later date, a mathematician trying to solve a problem should explore that wild idea and check the details later, or a student seriously interested in studying the arts or pure sciences should defer some of the technical aspects until a later date.

There are at least four reasons this principle is suggested by the model. The first is that it leads to the type of path discussed above. This principle clearly emphasizes the g parameter, the generalizing type of assimilation. A judgement of an idea is a careful examination of its details, which is an emphasis on parameter d. Thus, in this terminology, brainstorming is an exaggerated degree of deferred judgement. It is characterized by a path where the d parameter is temporarily set to a small value and the g parameter increases. Eventually the ideas must be evaluated, but since Ac is path dependent, its terminal image may be significantly different than what it would have been had the evaluation been performed at an earlier stage.

A second reason is that while Ac is path dependent, it is not everywhere continuous with respect to the choice of paths. Consequently, a path altered by evaluations at an early stage may lead to sharply different structures.

Third, an evaluation at an early stage, or a constant evaluation, uses up valuable space in the STM. Since this space is limited, all of the space should be used to attempt to solve the problem.

Fourth, the evaluation may be applied during the incubation period or during the critical stage of conflict near the threshold line leading to an incubation period of the second type. Consequently, instead of having a positive effect upon the problem, the evaluation may lead to a regressive method of problem solving.

3. Independence. This is another form of deferred judgement. When a nontrivial problem is encountered, it should be examined, studied, and at least a partial attempt be made to solve it before outside information is sought. For example, J. E. Littlewood (1968) recommends that when attempting to solve an unanswered research problem, you should understand the problem, but, at least initially, you should stay away from learning how other people have attempted to solve the problem. The fact that Ac has discontinuities with respect to paths shows why this is good advice. (I have always been impressed with the number of highly acclaimed people in the history of the arts and sciences who were, to some degree, self-taught. Therefore, we can assume their path in the d-g space was not the standard one.)

4. *Simplify the problem*. Simplify the problem until it assumes a form which is either trivial or solvable. Examine the solution. This may suggest a plan of attack. If it does not, then generalize this new problem and solve it. Essentially this is going back to first principles where it is easier to jump from one surface to another. Also, it aids STM, since the simpler problem does not have as many details.

5. *Reasoning by analogy*. There are two stages in the model which suggest this approach. The first is the initial assignment of a message—the problem may be simplified if it can be related to other problems, even from other fields, having a similar conflict. This shows the importance of a general formulation of the statement of a problem. In this fashion, the problem becomes a part of an already partially established structure. It is difficult to start a problem with d and g both small, that is, start the analysis of an event from scratch. However, if a problem can be attached to an existing structure, perhaps even from a different "field," a solution may be easier to obtain.

Second, reasoning by analogy helps increase the value of the g parameter. Since this method is an attempt to compare the message with problems already solved, or problems easier to solve, it is almost by definition the seeking of similarities between this given message and others.

6. *Extension*. Once a new technique or organizational structure has been developed, use it to interpret as much of the environment as possible. This is a well-established approach employed by successful researchers in all fields. It is the logical extension of step 7'.

Elsewhere other strategies will be discussed, but for our present purposes the above short list will suffice. It should be clear how these strategies are generated. The various stages and mechanisms of problem solving are examined to locate both their weak and strong points. A strategy is then proposed which will aid the former and emphasize the latter. For example, the limitations of STM suggest such external aids as checklists or matrices to show the different possible combinations of concepts. It also points out the importance of adopting a convenient notational system, etc.

Education

If teaching is the process of relating some structure which has been imposed upon the environment to the cognitive structures of the student, then a goal of education is to eventually eliminate the student's dependency upon a teacher or a second person in the interpretation of the environment. From this model, it is clear that before a student can achieve this state of independence and self-study, his education must provide him with four things: organizational structures, attitude, strategies, and motivation. If any of these are missing, the student will be handicapped in his dealing with a changing environment. The structures of today cannot be expected to suffice a decade from now. Let me briefly review these terms under different labels.

1. Content. If a message is to be interpreted, it must be assigned to some existing cognitive structure. For this to happen, the structures must exist. Thus, a formal educational program should provide courses which will develop structures necessary for the interpretation of future messages.

2. Confidence. An educational program must develop the student's confidence in his ability to interpret the environment. This is important in order to avoid the regressive method of problem solving. Confidence is needed to handle the conflict which appears in a nontrivial problem and the period preceding an incubation period of the second type.

Methods for instilling confidence are widely known, and reports of successful methods are the basis of some excellent popular books on education. Of course, the basic idea is to convince the student he can find or could have found the solution. For example, in the discovery method, this can be accomplished by providing gentle hints which suggest the obvious next step to the student in the sense it helps the student to formulate his or her own ideas. I suspect that without this judicious assistance the discovery method could lead to regressive methods of problem solving.

Because of this, I believe that while it is a compliment if students leave a lecture impressed with the beauty of the subject, it is even a higher accolade to the lecturer if the students feel cheated that they had not been born earlier because it is clear they would have discovered the theory.

3. Strategies. By example or by curriculum design, a student must be exposed to the techniques and approaches for handling new problems. This can be done when the student is introduced to the particular techniques of problem solving for a given subject, if these techniques are described in a general fashion. ("Let me see, where did we see that before?" "Maybe if we consider a simpler version we can see what's going on." "Hey, this seems to be similar to that problem in subject A; maybe there is a connection.") In other words, overly polished lectures, not to be confused with "carefully prepared," deprive the students of the opportunity to see the subject being developed. What is happening here is that such an approach trains the students how to use the two different types of assimilation.

4. Independent thinker. The above three traits will probably suffice to allow the student to react to different messages and changes in environment, but it will not permit him to become an original thinker. To do this, he must learn to generate and solve new problems on his own. Only after this is done, can he develop independent interpretations of the environment. However, the only way a person can become an original thinker is to do original thinking. This raises the question whether this can be taught, or at least directed. I believe the answer to be yes, modulo the patience of the instructor.

From the above, we see that perhaps the best way to stimulate original thinking is to start with a simple problem, solve it, generalize the problem, solve the new problem, etc. This provides movement for the path in the d-g space. But of greater importance, it provides movement for the path for a new subset of messages from the environment, a subset that has not been previously interpreted. Once d and g become well developed, i.e., they have large values, we can expect that new problems will be generated in an automatic fashion. However, note that the success of this approach presupposes a certain degree of development in the first three stages.

By reading their early published papers, it is clear this is the approach used by some successful researchers when they switch their field of interest. J. E. Littlewood suggests a similar approach in a recent book (1968), and R. P. Boas told me that G. H. Hardy used this approach to train his graduate students to do research. After a student solved several successively more general problems, the student would begin generating his own problems. While this example comes from graduate level education, there is no reason this approach cannot be applied at much earlier levels within the limits imposed by the degree of development of the first three stages. Indeed, since it involves the development of original interpretations of the environment rather than the acceptance of offered interpretations, this "research" should be started at the earliest possible age.

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Learning Difficulties in Children's Arithmetic:

A Clinical Cognitive Approach*

Herbert Ginsburg

Cornell University

For several years, we have been conducting research on the developmental psychology of children's mathematical thinking. Employing a cognitive approach, we have attempted to describe the intellectual processes children employ in their arithmetic work. This paper begins with a brief summary of our research and theory, and then presents some hypotheses concerning learning problems. Insight into these difficulties or learning problems may be gained through a clincial-cognitive approach to research. The approach is illustrated by two case studies. Finally, we consider implications of our work for research in mathematics education.

Theory and Hypotheses

Aim

The primary aim is to provide an account of the processes children use in doing school mathematics. The focus is on the child's understanding and misunderstanding; on his strategies; on the procedures that result in error as well as success. The research examines both the mathematical activities that children develop outside the school context and the formal knowledge of mathematics which children acquire in the academic setting.

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Method

The research is based mainly on the use of Piaget's clinical method (see Ginsburg & Opper, 1969) as applied to children from about 4 to 12. The clinical method begins with a specific task for the child. Often, the task is "concrete" in the sense of involving real objects, written problems, and the like. The child's responses to the task include verbal statements, written responses, and behavioral phenomena like counting on the fingers. The interviewer has the freedom to follow up on the child's responses (verbal or otherwise) if it seems important to do so. The clinical method is thus an extremely flexible (nonstandardized) procedure for investigating children's intellectual activities in response to specific and often concrete problems.

The method is based on some sound theoretical principles. The chief of these is that intellectual activities--"underlying processes" or "cognitive processes"--are complex, and their measurement should display a comparable degree of complexity. Since a child may solve an addition problem in curious and complicated ways, the measurement technique, to be effective, must display corresponding subtlety. The clinical interview attempts to follow the child's arguments, to challenge them, to pose new problems, etc. The attempt to measure underlying thought processes demands the use of flexible measurement procedures.

A second theoretical principle which requires use of the clinical method or an equivalent is the distinction between the behavior a child may show in some particular situation (his performance) and what he "really knows," or "the best he can do" (his competence). The idea is simple. The child's behavior in a testing situation may be influenced by a variety of factors: his true intellectual competence, fatigue, boredom, test anxiety, etc. We are interested not in his test behavior per se, but in his test behavior as an index of underlying intellectual competence. We recognize that the child's overt test behavior (performance) may not always reflect the intellectual processes which represent the best he can do (competence) but may instead be depressed by other (e.g., motivational) factors.

Standardized testing procedures, despite superficial attempts at rapport ("Now, children, we are going to play a game"), often fail at uncovering competence. See, for example, Labov's (1972) account of how standard language tests fail to reveal black children's linguistic competence, whereas flexible procedures of various types are far more successful at this. Consequently, one must use methods like the clincial interview in an attempt to measure underlying competence. For a fuller discussion of this point, see Ginsburg and Koslowski (1976).

The Theory

A review of the literature and our own research (Ginsburg, in press-a, in press-b) suggests that children's knowledge of arithmetic[1] may be conceptualized in terms of three cognitive systems which may operate concurrently as the child solves problems. System 1 involves patterns of perception and thought which are used to deal with quantitative problems but do not employ counting or other explicit forms of mathematics. System 1 develops outside the formal school setting and hence may be termed informal. Further, since System 1 does not involve counting or other specific information or techniques transmitted by culture, it may be termed natural.

System 2 involves counting and related procedures by which children cope with quantitative problems in the absence of formal instruction. System 2 is thus informal insofar as it develops outside the context of schooling but is cultural since it depends on the social transmission of counting.

System 3 involves techniques for dealing with symbolic, codified arithmetic. These techniques typically develop in the school context and hence may be termed formal; they are products of the culture and transmitted by it.

In individual children, the three systems may exist in relative isolation from one another or may display some degree of integration. Consider each system in turn.

System 1. By the age of about 4 or 5 years, the young child can easily perceive which of two randomly arrayed sets (see Figure 1) has

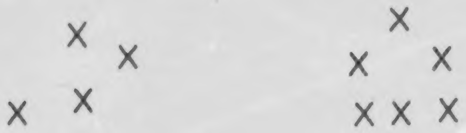

Figure 1

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[1] Note that our work is limited, for the present, to the study of arithmetic, not geometry, etc.

"more" than the other, at least when relatively small numbers of elements are involved. The child need not use counting to solve the task. Typically this method is based on the perception of area. That set which occupies more space is considered to have the greater number. The child's method is frequently successful because, all things begin equal, the set occupying the larger area does indeed have the greater number. Area occupied is correlated (imperfectly) with numerosity.

There are several points to stress concerning System 1:

1. Before entering school, or outside the context of formal education, children and adults develop many techniques, like the one described, for solving quantitative problems which do not demand a numerical response. Children can deal with one-to-one correspondence, equivalence, and series. Their techniques are reasonably effective and can serve as a basis, even a prerequisite, for later school mathematics.

2. So far as is known, elementary informal techniques are cognitive universals. All children seem to develop them.

3. Usually, Piaget's theory is interpreted as proposing that young children exhibit many deficiencies with respect to informal mathematics. That interpretation is only partly accurate. It is true that the young child cannot conserve numerical equivalence, perhaps because of immature thought processes, as Piaget proposes. But one must keep in mind that the young child does not fail at all aspects of informal mathematics. Indeed, there are many facets of the young child's informal thought which are quite sophisticated and powerful, as Piaget himself asserts.

System 2. At some point in history man invented counting, a technology permitting important advances in cognition. Counting allowed those using it to solve quantitative problems with a new precision. The child learns to employ counting too, and it permits important intellectual advances.

During the preschool years, the child begins to say the counting numbers and to count objects. His counting is informal; in general, it develops outside the context of formal education. During the elementary school years, the child expands the range of his counting activities: He learns to say larger numbers, to count greater numbers of objects, and to enumerate more efficiently. He also blends counting with formal arithmetic. Counting is no longer purely informal. At the same time, counting is very much the child's preferred method. He finds it comfortable and uses it to solve various problems in arithmetic. Indeed, the bulk of the young child's arithmetic may involve counting. During the elementary school years, the child may attempt to apply counting to addition, subtraction, and other arithmetic problems. Furthermore, here is a conjecture: Probably the great majority of young children interpret arithmetic as counting, regardless of how they are taught. Whether they are taught sets or number lines or logic, new math or old math, they probably use counting as the basic method for dealing with arithmetic.

Consider the child's use of counting for work with real objects. The interviewer asked Kathy, a second grader, how she would add 9 and 6 dots.

> K: I would draw a box and put the dots in and count them.
> I: Know any other ways?
> K: Well, I could count on my hands, like 9, 10, 11, 12, 13, 14, 15.
> I: And you think that will give you the total number of dots in the boxes?
> K: Yes. 'Cause I used the biggest number first so I don't have to count as much.

Counting can be a powerful tool for doing arithmetic. Some "primitive" cultures have developed rather elaborate systems of arithmetic based almost entirely on counting. Rosin (1973) refers to the fact that in rural India illiterate persons can deal effectively with money lenders and shopkeepers who are skilled in the traditional arts of calculation. Rosin analyzes the calculation executed by an illiterate person who has devised an arithmetic of his own. The informant can solve problems that require the use of whole numbers and fractions. His methods are based heavily on elementary counting. For example, Rosin notes that "the operation of addition is accomplished by counting one number onto another" (p. 5). This is done by a highly elaborate system of finger and joint counting. Furthermore,

> other arithmetic operations, such as multiplication, doubling, halving, and quartering are worked out by memorizing results obtained by counting the finger joints. Each of these operations are painstakingly learned, first through calculation and then memorization. Learning these operations is worthwhile because they aid in a variety of customary activities. (p. 5)

In brief, extensive arithmetic systems can be derived from elementary counting and addition. The culture and the child may go a long way with these skills.

Consider now a few general comments concerning the nature and development of counting.

1. Typically, the child's counting is an organized activity. For example, adding two numbers often involves an organized process like beginning with the larger number and counting on. Children's arithmetic, even on the most elementary level, does not involve only memorized facts or rote activities.

2. The child uses counting to assimilate arithmetic problems. Thus, the child may add by counting, or may subtract by counting. In Piaget's language, children tend to assimilate new problems into already existing schemes. Children seem confident in and comfortable with counting. They revert to it when other procedures fail.

3. Counting is a technique of considerable power. History suggests that arithmetic developed from counting. Anthropological studies show that some illiterate cultures use counting to perform difficult calculations. Many cultures seem to place heavy reliance on counting as the basic arithmetic technique. And children employ counting as a technique for the solution of various arithmetic problems. Procedures of this type--for example, finger counting--can be remarkably effective.

System 3. The child who attends school is exposed to more than counting. He is taught symbolic arithmetic. He encounters written symbolism, algorithms, and mathematical principles. These cultural inventions and discoveries are more powerful than counting and, if used properly, can provide the child with greater efficiency in quantitative problem solving. Consider now how the child deals with symbolic arithmetic.

Algorithms. A good deal of elementary school education is devoted to teaching the child to do addition, subtraction, multiplication, and division with whole numbers. Typically, the teacher shows the child a standard algorithm for calculation (e.g., the borrowing or regrouping method for subtraction), and the child is expected to learn it. Often this happens and the child does arithmetic in the standard ways. There is little to say about this except that the successful use of a standard algorithm does not necessarily imply any kind of understanding. The child may add a column of numbers by the conventional rule without knowing the rationale for that procedure. In brief, it is evident that children often calculate in the ways they are taught, but understand little about them.

Invented procedures. While common, standard algorithms are not the only methods which children use to calculate. Much of children's computation involves invented procedures--methods which in part the child devises for himself and which in part are based on school learning. Some invented strategies may be characterized by their use of previous knowledge or techniques, that is, by the assimilation of current problems into existing schemes. Addition by counting is one example of such an invented strategy.

Other invented methods seem to use combinations of procedures, some of which may or may not have been learned in school. Carol, at 8 years 7 months of age, was asked to perform some simple word problems. The interviewer began by establishing that Carol knew that there were seven days in a week and 24 hours in a day. Then he asked, "how many hours in a week?" and, "what do you do to get the answer?" Carol did not conceptualize the problem in terms of multiplication. She replied, "just add them up," and wrote the following column of numbers.

$$\begin{array}{r} 24 \\ 24 \\ 24 \\ 24 \\ 24 \\ 24 \\ \underline{24} \end{array}$$

Then she used a combination of procedures to solve the problem. She began by adding the first four 4's in the column.

C: 4 and 4 is 8. 8 and 8 is 16.

Note that Carol added the first two 4's to get 8; implicitly, she added the second two 4's to get 8 and then added the first two sums to get 16.

Then she continued.

C: 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28. That's the 28.

So after adding 8 to get 16, Carol used a counting procedure to reach 28.

Next she employed a different method.

C: I put the 8 down and then I carried the 2 up there. I brought it upstairs.

$$\begin{array}{r} {}^{2}24 \\ 24 \\ 24 \\ 24 \\ 24 \\ 24 \\ \underline{24} \\ 8 \end{array}$$

C: Then all I have to do is . . . 8 x 2 . . .[she wrote down 16] . . .168.

So Carol finished the problem by multiplying.

Instead of employing the conventional method of solution which she had been taught, Carol assimilated the problem into her own conceptual framework. She used a rather unique strategy, involving a variety of computational methods: First she regrouped and added, then she counted, then she carried, and finally she multiplied. All this to find the sum of a column of numbers!

Errors

Children often make mistakes in arithmetic. While the facts of failure are obvious, their causes are not. Some propose that the child's failure is due to "deficient intelligence" or to "low mathematical aptitude" or "learning disability" or even to "cultural deprivation." We believe that explanations of this type are overly vague, imprecise, usually not helpful, and sometimes even dangerous. To explain the child's failure one must examine his thought processes in some detail; one must attempt to discover those mental procedures which underlie his mistakes. Research shows that the child's failure is often the result of an organized procedure.

For example, Joe, 11 years of age, in grade 5 was presented with the following addition problem:

```
     14
     37
     7
   3406
  14526
     98
  _____
```

and Joe's response was simply to add "downward." Six and six is 12, carry the one, etc. It is evident that Joe's mistake on this problem was the result of his blind application of the usual addition algorithm to a situation where it is inappropriate. Joe neglected to rearrange the numbers in the problem presented to him. Joe's mistake was not simply the result of random error, lack of memory for the addition facts, or even "low intelligence" (whatever that is). Rather his error was the result of a systematic misapplication of a correct procedure. Later the interviewer asked Joe to write down some numbers for an addition problem: 19; 472; 3; 6,023; 71,845; 56.

Joe wrote:

```
 19
 472
 3
 6023
 71,845
+56
 __________
```

Joe's mistake was obviously to line up the numbers from left to right rather than right to left. Again we see that Joe's mistake is not capricious. Rather it is a result of a systematic application of an incorrect rule. Other research (e.g., Erlwanger, 1973) confirms these observations: Errors often result from systematic strategies.

Consider now a few general principles concerning System 3:

1. Like counting, children's formal arithmetic is typically permeated by rule-governed organized activities. Even so simple an act as writing a number operates by rule. More complex activities, like calculating a sum, are characterized by a variety of underlying organizations.

2. Children's arithmetic is often based on ontogenetically prior schemes and counting is the chief of these. Children add by counting and multiply by adding. In general, they assimilate new problems into familiar ways of doing things.

3. Children often solve arithmetic problems by invented procedures. Children do not simply employ standard algorithms as taught in school; instead they devise their own procedures. These usually rely in part on assimilation into familiar schemes. This seems to be the way in which children gradually learn written arithmetic, which of course can be more powerful than oral counting methods.

4. Children's mistakes are not capricious or the result of low "intelligence" or "mathematical aptitude;" they are the products of strategies.

Relations among the systems. Frequently, the three systems operate concurrently in the individual child. Often, however, there may be discontinuities among or within the systems. One major kind of discontinuity involves formal written algorithms and other procedures (e.g., mental calculation, invented procedures, counting methods).

Churchill (1961) reports this observation concerning Caroline at 6 years of age. The teacher asked Caroline: "If you bought 24 bulbs and six of them were tulips, how many daffodils would you have?"

Caroline responded correctly, "eighteen."

"Now you can write down what you have done in your head?" Caroline wrote:

$$\begin{array}{c} 6 \\ \underline{24} \end{array}$$

Then:

$$\begin{array}{r} 6 \\ -\underline{24} \\ 22 \end{array}$$

She then said, "But it ought to be eighteen, oughtn't it?"

Caroline could do a simple subtraction word problem in her head, but she was not able to calculate it on paper. Further, she believed that the mental subtraction gives the correct result, as indeed it does. There is a discrepancy between the child's solving the problem in one mode (mental calculation) and another (written calculation). Children seem to have particular difficulty with the latter; they often cannot use standard algorithms to calculate on paper while at the same time they can solve essentially the same problems via alternative (nonwritten) procedures.

Why does the child have so much trouble with written work? Before encountering written mathematics, most, if not all, children invent sensible methods for dealing with arithmetic problems in the real world. Then written symbolism--those strange marks on paper--is introduced, and children need help in interpreting it. They need to see the connection between what they can already do and the arbitrary representations. They have to learn the meaning of symbols--that is, how symbolism relates to previously developed knowledge.

If discontinuity among systems implies misunderstanding, the integration of systems involves understanding. Thus, one way in which children "understand" a calculation is to connect it with or assimilate it into a more elementary calculation procedure like counting. For example, on the most elementary level, Seslie was asked why she wrote 6 + 3 = 9 on paper.

> S: This is 6 + 3 = 9 and I put a 9 here, just remembering. Some people say 6 + 3 = 8: They get their answers wrong. But 6 + 3 is 9 'cause you can tell. . .adding 3 more is 9. . .6, 7, 8, 9. [Counting on her fingers.]

Seslie knew that 6 + 3 = 9 because she could make a connection between this addition fact and her counting scheme. She interpreted addition in terms of counting; she understood the former in terms of the latter.

To summarize:

1. There are often discontinuities among or within systems--particularly involving the child's written mathematics and other areas of his thinking.

2. Understanding may be considered a comfortable integration between what the child must learn (usually some aspect of symbolic mathematics) and what he already knows (often an informal or invented procedure).

Implications For Learning Problems

While deriving from the study of relatively "normal" children, the theory described above seems to have implications for the understanding of learning difficulties in mathematics. Consider three propositions concerning such difficulties and their study.

1. According to the theory outlined above, learning difficulties in mathematics (or any other area) have a systematic basis in intellectual processes. Learning difficulties result from orderly rules which produce error, and involve gaps between powerful informal knowledge or invented procedures and faulty written algorithms.

A corollary is that errors are not capricious. Nor does it seem useful to propose that they stem from mental entities like deficient "intelligence" or low "mathematical aptitude." Concepts like these are both vague and impractical. No one has a clear theory of intelligence or mathematical aptitude. Measurement of either of these entities suggests nothing about how remediation might procede.

2. There seems to be a clear basis for the remediation of learning difficulties. Many, of not all, children possess relatively powerful informal knowledge or invented strategies which may be used as a basis for learning school mathematics. For example, before entering school, children can use counting for rudimentary calculation. Such abilities are intellectual assets for child and teacher, both of whom can use the child's informal knowledge as a foundation on which to build a sound understanding of school mathematics. A focus on the real abilities which children bring with them to the task of coping with school arithmetic is especially important for helping poor children, minority children, and children with learning difficulties.

3. The clinical interview is an important tool for the study of learning difficulties. One reason is that the clinical interview procedure (described earlier) is based on a sound theoretical rationale. The clinical interview is not a preliminary or sloppy procedure which needs to be standardized; it is a legitimate method in its own right. A second reason is that the clinical method may help overcome the usual difficulties associated with standard assessment procedures. There is now a research literature (e.g., Cicourel, Jennings, Jennings, Leiter, MacKay, Mehan, & Roth, 1974) documenting the difficulties children have with standard tests--e.g., how children misinterpret the tests, are not motivated to take them, and how the tests do not measure what they claim to. Anyone who has taken such tests knows, or should know, how bad they are. The clinical interview may overcome many of these difficulties. We have found it useful in dealing with children who do not test well by other means (e.g., the case of Peter, in Ginsburg, 1972).

Unanswered Questions

These principles require further investigation. One reason, as pointed out above, is that they derive from the study of "normal" children experiencing relatively minor difficulties in their arithmetic work. This raises the question of the extent to which the principles apply to children exhibiting severe difficulties in mathematical thinking. For example, we need to know whether children with "learning disabilities" nevertheless possess a relatively powerful system of informal knowledge. Similarly, we need to know whether children diagnosed as suffering from neurological deficits make errors which result from systematic strategies and whether such children exhibit gaps between adequate informal or invented procedures and faulty written algorithms.

A second reason for further investigation is that the principles are crude and based on a relatively small body of data. Consequently, many questions remain open.

1. We need to have a clearer understanding of the intellectual processes producing errors. Is it possible to develop a taxonomy of the major types of processes leading to error?

2. We need a more detailed portrait of children's intellectual assets. Relatively little is known of the informal strategies which children acquire outside of school for the purpose of calculation.

3. We need further investigation of the notion that there exist gaps between a child's informal knowledge and his school learning. What makes written symbolic arithmetic so difficult for children to assimilate?

4. While there appears to be a sound rationale for the clinical method, and while some have used it with considerable success, we know relatively little about the clinical interviewer's mode of operation and its strengths and weaknesses. What is the interviewer's strategy and how effective is it?

One method for getting insight into these issues is the clinical cognitive case study.

The Clinical Cognitive Case Study

Description

The clinical cognitive case study involves the use of flexible methods, particularly the clinical interview, for the purpose of investigating the intellectual roots of learning difficulties in individual children. The approach is clinical in two senses: It employs the clincial interview method, and it is concerned with the remediation of severe learning difficulties. The approach is cognitive in that it attempts to identify cognitive processes which cause the difficulty. The approach involves clincial case study in that it deals with the long-term and intensive investigation and treatment of individuals.

The clinical-cognitive case study method may be fruitfully employed to answer the questions posed above. It has several advantages:

1. It permits a test of our theory of arithmetical learning difficulties on a population which exhibits them in extreme form. The study of "normal" children may not be as useful from this point of view.

2. The clinical method permits the discovery of new phenomena and, hence, may result in useful modifications and expansions of the theory. Thus, the clinical approach may make it possible to identify new intellectual sources of error in arithmetic. Standard tests do not seem to facilitate this kind of discovery.

3. The case study method permits an examination of cognitive complexity within the individual. For example, one may investigate the unique pattern of discontinuities among different aspects of the child's knowledge. Standard experimental designs often have difficulty in dealing with the "idiographic" (Allport's term).

4. Case studies of the type described provide an opportunity for the study of the clinical interview method itself.

Method

Our first case studies were done as follows. We went to a local Ithaca elementary school serving both middle- and lower-class children and asked a teacher of a combined third- and fourth-grade class for those students who were having the most difficulties with mathematics. We wanted to know nothing about the children except that they were having difficulties. Nevertheless, the teacher could not keep from telling us that all the children suffered from "perceptual problems." At the time of testing we knew little about the kind of instruction the children received except that to some extent they were using Suppes' Sets and Numbers text and were participating in the IMS programmed learning system. We interviewed each child once a week, over a period of four to six weeks. Most interviews were recorded on TV tape in a room adjoining the classroom. At the outset of the interviews, we demonstrated the TV taping to each child so that he or she would be familiar with it. After the first few minutes, the children seemed to ignore the TV camera which was in full view throughout all sessions. Consider first an interview with Patty.

Patty

Patty was first given a subtraction problem $\begin{array}{r} 962 \\ -439 \\ \hline \end{array}$ which she did correctly using the standard algorithm with "borrowing." She wrote: $\begin{array}{ccc} 9 & 6 & 12 \\ 4 & 3 & 9 \\ \hline 5 & 2 & 3 \end{array}$ Then she did an addition problem $\begin{array}{r} 226 \\ +421 \\ \hline 647 \end{array}$. She followed the standard algorithm except that her procedure was to count out loud and on her fingers when she could not remember the relevant number facts. Thus, she added from right to left, remembering that 2 and 2 are four and counting to determine that 4 and 2 are 6. Next she did $\begin{array}{r} 29 \\ +\ 39 \\ \hline 68 \end{array}$. Again this involved the standard algorithm, with carrying, and was done in part by counting on the fingers. Patty would give no rationale for the carrying of the 1. All she could say was that it was wrong not to carry the 1 and instead place it on the bottom with the 6 and the 8.

These first few incidents reveal some basic things about Patty. First, she was familiar with the common borrowing and carrying algorithms for subtraction and addition, respectively. Second, she could execute these fairly smoothly, at least under certain conditions -- specifically, when relatively small numbers, each having the same number of digits, were involved, and when she could count on her fingers and therefore did not have to rely on memory for number facts. Third, she did not seem to know much about the theory of place value and hence could not explain why one carries, although she could do it. So far it does not seem as if Patty had any particular difficulties -- perceptual or otherwise -- with arithmetic.

Then we have the following exchange:

Interviewer 1 (Barbara Allardice): I'm going to give you another problem. You seem to be doing pretty well adding. Suppose you have 29 again and 4.

Patty wrote $\begin{array}{r} 29 \\ +\ 4 \\ \hline 69 \end{array}$

I-1: What does it say right here?
P: 29 and 4
I-1: Are how much?
P: 69

This then was Patty's first error in the interview. She got a wrong result (69) because she employed a wrong strategy. At this point in the interview, a sensible hypothesis concerning the strategy is as follows: When the addends have unequal numbers of digits, she lines them up from left to right and then applies the standard addition algorithm, with counting on the fingers, from right to left. The initial assessment then was that Patty had a systematic but incorrect strategy which leads to error.

At this point the teacher might wish to intervene and straighten out Patty's incorrect method. But subsequent portions of the interview shed a different light on some of Patty's abilities.

I-1: Are you sure that 29 and 4 are 69?
P: Altogether?
I-1: Uh uh [yes]
P: No
I-1: How much are 29 and 4?

Next Patty made a large number of tallies on the bottom of the page:

| | | | | | | | | etc.

She appeared to count them, at least sometimes using her fingers. Then she announced the result: 33.

Clinical interviewing is a highly theoretical activity in which the interviewer continually invents hypotheses and tests them. At this point the interviewer's hypothesis--that is, her theory or assessment of Patty--was something like this: Patty has an incorrect strategy for written addition of the type descrived above. At the same time she has an effective strategy for performing addition when real objects--here tallies--are involved. The correct strategy for objects is essentially to combine the two groups and count the aggregate. There is a discontinuity of the type described earlier between written work and arithmetic with real objects.

We see that so far the assessment has depended heavily on the theoretical framework eleborated above. We have analyzed Patty's arithmetic performance in terms of systematic formal processes which lead to errors (the written algorithm); systematic informal processes (combining and counting objects) which lead to successes; and the discontinuity between the two.

The interviewer now tried to determine whether Patty placed more confidence in the written procedure than in the counting one.

I-1: 33. O.K. How come this says 69 [pointing to the written work]?

P: Ooops! Because you're not doing it like that [pointing to the tallies]. Oh, this is wrong.

Apparently Patty saw that her answer of 69 was wrong, and that it differed from the result obtained by counting. She changed the 69 to 33. Patty seemed to have greater confidence in the result obtained from counting than from written addition.

At this point the interviewer decided to challenge Patty's new response (33). The interviewer offered a counter-suggestion to see how firm was Patty's belief in her counting-derived result. Piaget often uses such a counter-suggestion to test the child's confidence in a verbalization.

I-1: How can you put a 3 here [referring to the second 3 in 33] if it says 9 here [referring to the 9 in 29]?

Patty looked at what was written-- $\begin{array}{r} 29 \\ +4 \\ \hline 33 \end{array}$ --and changed it back to $\begin{array}{r} 29 \\ +4 \\ \hline 69 \end{array}$.

P: That's 9 and that's gotta be 6. It's just that you're doing it differently than that.
I-1: So you get a different answer.
P: Yeah. 'Cause you're adding all of this up together [meaning the tallies].
You're not adding it all up altogether this way [pointing to the written work]. You're putting the 9 down here and the 6 down here. 'Cause you're adding 2, 4 is 6 and 9 by itself and that's 69.
I-1: So when you do it on paper you get 69 and if you do it with the little marks you get how many?
P: 33. Because you're adding all of it altogether. And you're not doing it over here.
I-1: Suppose we had 29 of these little chips and put out 4 more. Would we get 33 or 69?
P: 33.
I-1: How do you know?
P: Because I did it down here and I added 4 more onto it [points to the tallies on the bottom of the page].
I-1: O.K. So that means these chips would be like these lines.
P: Yes.
I-1: What would be another thing that would be like this [the written problem] where I could get 69?
P: There ain't no way, I don't think.

So Patty knew that several ways of counting objects (tallies, chips) were equivalent but could think of nothing similar to the written problem.

I-1: Let's see. You have 29 and 4 and you get 69. Suppose you had 30, so you had 1 more here, and 3, so you have 1 less there. Would you still get 69?

The interviewer's intention was to present Patty with a situation producing a contradiction. 30 + 3 should yield the same sum as 29 + 4. 30 had 1 more than 29 but 3 had less than 4. Yet by Patty's method she should get a different result for $\begin{array}{r}29\\ +4\\ \hline\end{array}$ than $\begin{array}{r}30\\ +3\\ \hline\end{array}$. Would Patty see the contradiction?

P: No. 'Cause you'd get bigger than 69.

She did 30 + 4 instead of 30 + 3 and wrote: $\begin{array}{r}30\\ +4\\ \hline 70\end{array}$

P: Yep. I told you you'd get more than 69.

Up to this point we seem to have evidence to support the hypothesis that Patty uses a combining and counting method to get correct sums when real objects are involved and that she uses an incorrect algorithm--lining up on the left with uneven numbers of digits--when written numbers are involved. The faulty algorithm is used in at least two cases.

At this point a second interviewer (the present writer) who had been observing the interaction wanted to test the generality of Patty's written algorithm. How would Patty react to extreme cases?

I-2: O.K. Let me try something, Patty. Can you write down for me 100 + 1?

Patty wrote $\begin{array}{l}100\\ +1\\ \hline 200\end{array}$ and said: Zero, zero, and two. It would be two hundred.

I-2: 100 plus 1, huh? Do you think that's right? Got any other way of doing it?
P: No. Unless the one is on the wrong side. Unless the one's supposed to be there [points to the one's column].
I-2: Where's the one supposed to be?
P: I think it's supposed to be there [points to the hundred's column].
I-2: You think it's supposed to be there, huh? O.K. Let's do another one. What about 10 plus 1?

Patty wrote $\begin{array}{l}10\\ +1\\ \hline 20\end{array}$ and said: That's zero and that's 2. 20.

I-2: 20, think that's right?
P: Yeah.
I-2: Got any other way of doing it?

Patty indicated no.

At this point it was clear that Patty's written method generalized widely. Now the interviewer wanted to get Patty to see the discrepancy between her written and counting methods.

I-2: Well, suppose you couldn't use paper at all and I said how much is 10 plus 1?
P: I'd count on my fingers.
I-2: Why don't you do it?

Patty held up 10 fingers and stared at them.

P: You have 10 [looking at the fingers]. You put the zero on the bottom [draws a zero with her finger].
I-2: Just use your fingers now.
P: Then you put 2 and you add 1 and 1 and it's 2.

Patty seemed unable to count 10 and 1 on her fingers. Instead she perseverated in using the written procedure, apparently doing "in her head" something very much like

$$\begin{array}{r}10\\ +1\\ \hline 20\end{array}$$

I-2: What about on your fingers? Show me how you do it on your fingers. You can use my fingers too [I-2 puts both hands on table]. Put out your fingers too.
P: You put the zero on.
I-2: No, I don't see any zeros. All I see are these little fingers. Never mind zeros.
P. That's hard. [looks as though thinking intently]
I-2: [to I-1] Barbara, put out your fingers, too. Now you have all kinds of fingers to work with, Patty. Now you figure out how much is 10 plus one.
P: You have to put a zero underneath.
I-2: I don't see any zero at all. All I see are these fingers.
P: O.K. If you want zero you have to take those ten away [points to I-2's fingers]. You put zero, then you have 1 and 1 left and you add them up and you get 2. So it's 20.
I-2: Can you do it without zeros?
P: No.

Patty's perseveration was very strong. She could not seem to get away from using the incorrect algorithm.

I-1: How about with little marks on your paper like you did here? How can you make 10 and 1 on the paper?

In other words, could Patty use tallies to solve the problem of 10 and 1? Patty made 10 tallies.

P: [whispering] 1, 2 . . . 10 and then put one [makes another mark].
I-1: How many do you have altogether?
P: Eleven

She made a sweeping motion with her hand as if to indicate that she meant to combine the two sets.

I-1: Eleven?
P: Yeah. Altogether.

This incident seems to support the hypothesis that Patty uses one procedure (the incorrect algorithm) for written numbers and another procedure (combining and counting) for real objects (including tallies). But consider the following episode.

I-2: Eleven altogether. What about . . . Let's do this . . . We've got...There are 10 of these [chips] and here's one more. How many do you think there are altogether?
P: Altogether, it would be 11.
I-2: O.K. What about 10 plus one, not altogether, but plus?
P: Then you'd have to put 20.
I-2: Then you'd have 20, I see. What if we write down on paper, here's 20, now I write down another 1, and you want to find out how much the 20 and the 1 are altogether.

The interviewer wrote 20 1 placing the numbers side by side.

P: It's 21.
I-2: O.K., now what would 20 plus 1 be?
P: Twenty plus one? She wrote

$$\begin{array}{r} 20 \\ +1 \\ \hline 30 \end{array}$$

This behavior indicates that the original hypothesis was wrong. She does not just use the counting strategy with real objects and use the written algorithm with numerals. Matters are more complex. Perhaps we can state a new hypothesis as follows. The crucial distinction is not so much between numerals and real objects as between the word "altogether" with the strategy it elicits and the word "plus" with its strategy. "Altogether" seems to elicit the strategy of combining real objects and counting them, or counting on when numerals are involved. "Plus" seems to elicit the incorrect addition algorithm when both numerals and real objects are involved. "Altogether" seems to be the child's natural word--used in her everyday life--which is associated with an informal strategy used in everyday life, viz., combining and counting. By contrast, "plus" is a school word, associated with a formal algorithm which happens to be wrong.

While this hypothesis seems reasonable, further evidence is necessary to test it. The interviewer would need to determine if counting on is really used with numerals, if the strategies generalize to different kinds of numbers, etc.

This in the first interview with Patty. It teaches us several things.

1. Patty's main mistakes in addition are the result of a systematic strategy. This confirms our hypothesis that errors generally result from organized patterns of thought.

2. Patty shows important strengths. In particular, she can use a sensible strategy--combining and counting--to do addition. She shows no evidence of severe problems of any sort, including perceptual problems.

3. Our notion of a gap between different systems--Patty's incorrect algorithm and her counting and combining strategy--seems useful.

4. At the same time, the case study suggests that the theory needs to be expanded so as to include heavier concentration on linguistic factors. In Patty's case, the discontinuity was not between approaches to real objects and approaches to written work. Rather, for Patty, different words elicited different strategies. The case study thus suggests a comparison of the child's everyday mathematical words with those taught in school.

5. In general, the clinical case study method seems promising. It seems to have the flexibility to yield a rich portrait of the individual child experiencing learning difficulties. This portrait seems to do justice to the complexity of mathematical thinking and provide insight into both strengths and weaknesses. One measure of its success is that the case study yields information which suggests concrete teaching strategies. Thus, the teacher might choose to help Patty with addition by building on her strength--the counting and combining strategy. Instead of merely telling Patty to line up numbers differently, the teacher might help her to see the relation between her "altogether strategy" and the written algorithm. Then perhaps she can learn to see how and why one needs to line up numbers properly in the algorithm.

6. The case study illustrates a basic feature of the clinical interview method. The clinical interview is a hypothesis testing procedure. It is analogous to a series of experiments in the case of one child. Thus, to test the hypothesis that the causative factor is language ("plus" vs. "altogether") and not type of problem (real vs. written), the interviewer held type of problem constant and varied language. Thus, he did real object "plus" vs. "altogether," and then written number "plus" vs. "altogether." This is equivalent to a 2 x 2 factorial design, with of course only one subject. The clinical method thus attempts to manipulate the independent variable so as to produce results which allow reasonable inferences concerning various hypotheses.

Stacy. Consider now the case of Stacy, also a third grader, whose behavior illustrates the power of the clinical interview technique to identify intellectual strengths in a child who seems almost retarded at the outset. In the first session, I was the interviewer.

I: Can you tell me first what kind of work you are doing in math now?

Stacy responded, and continued to respond throughout the session, in a slow, quiet voice. Her manner was extremely diffident--even lethargic and depressed.

S: Lots of things.
I: You write stuff on paper. Can you show me what stuff you write?
S: Papers like math [she then began to write a sentence, not just numerals].
I: Can you read that?
S: Jimmy had 8 cats; he gave Brian 2 cats.
I: What comes next?
S: How many does Jimmy have?
I: How many do you think he has?
S: 5.
I: How did you know that?
S: He had 8, and 2 and 1.

It was very hard to get Stacy to respond--to indicate how she had done the problem.

I: How did you do that, Stacy? He had 8 cats, and he gave Brian 2 cats. How many did he have left?
S: 5. He had 5 and 2 others got away.
I: So how many did he have left?
S: 8 cats and I count back.
I: How do you count back?
S: 8 and I got 3 more and then I took 2 away.
I: What do you mean 3 more? Let's start from the beginning. Show me how you count back.
S: 8, 7, 6, 5, 4. He had 4 left.
I: How did you know to stop at 4? You went 8, 7, 6, 5, 4. How did you know to stop at 4?
S: Because there's 7.

This initial episode gives the flavor of the interaction with Stacy. She posed herself a very simple problem with which a third grader ought to have no difficulty. Indeed, the problem was in words, rather than written numerals, and involved a simple story: If Jimmy had 8 cats and gave away 2, how many would be left? In response to this problem, Stacy did several things. The most obvious is that she gave several different wrong answers. She changed her response several times. She indicated that her method of solution was by counting backward. But her behavior did not seem to be a simple product of this or any other strategy. Indeed, her responses were disorganized and chaotic; it is hard to see how any underlying organization could have produced them. In brief, the initial episode suggested that Stacy gave wrong answers to extremely simple problems, and that she seemed to have no organized method for producing answers.

The remainder of the first interview showed that Stacy's work was on an extremely low level. The interviewer then gave her a very simple problem. If there were 4 dogs and 2 ran away, how many would be left? Stacy gave the correct answer. When asked how she did it, she replied "Because 2 and 2 is 4." So Stacy seemed to do subtraction by remembering some relevant addition facts. Asked to solve this problem by counting backwards, she could not do so; she merely persevered in the addition or produced apparently choatic, senseless behavior.

Next the interviewer gave Stacy some simple addition problems. First, How much are 3 apples and 4 apples? Stacy answered, six, "because 3 and 4 is 6." Thus, the wrong answer is apparently the result of faulty memory of the addition facts. Asked to do the problem by counting. Stacy merely shrugged her shoulder and shook her head--behavior which she often displayed when she did not know what to do. Next the interviewer asked Stacy: "How much are 2 oranges and one more? She got the answer right, apparently because she remembered the number facts. That was the end of the first interview.

The initial results suggested that Stacy could do very little. She was struggling with problems which should have been trivially simple for a child her age. About all she could do was occasionally remember some number facts. She seemed unable to use counting procedures--which, as we have seen, are usually children's method of preference.

After the first interview, we were very discouraged. We seemed to have encountered a child--the first we had seen--who had almost nothing going for her. The initial results led us to formulate the following questions: Is she retarded? Can she hear properly? Was she very nervous or intimidated by the interviewer? Can she conserve? Can she count? How would she do with concrete objects?

Note that some of the questions refer to her motivation and some to her cognitive abilities. We wanted to know essentially whether the testing situation interfered with her true competence or whether she had much of any competence to begin with. Usually we assume that elementary school children's difficulty with mathematics is not due to inadequacy at the Piagetian stage of concrete operations. But Stacy did so badly that we could not assume this. Similarly, we could not assume that Stacy knew the counting numbers up to a reasonable limit.

Technical difficulties prevented the videotaping of the second interview, conducted by Barbara Allardice. ("Technical difficulties" means that the TV broke down, which often happens.) Nevertheless, Allardice reports that Stacy had no difficulty in conserving number nor in counting up to at least 80. There is some evidence then that Stacy had available some fundamental cognitive tools. She was probably in the concrete operational period, and she had reasonable knowledge of the counting numbers.

The third interview, also conducted by Barbara Allardice, was devoted to discovering what Stacy could do with real objects. The interviewer first asked Stacy to get 7 chips from a larger pile. Stacy took 7, one at a time, and put them in a straight line. The interviewer then asked her to get 3 more. Stacy did so, putting them in a line under the first as shown.

X X X X X X X

X X X

I: O.K. How many do you have altogether now?
S: Ten.
I: Ten. Very good. How did you figure that out?
S: Just counted them.
I: Counted them up. O.K. Now suppose we have one more. Can you get one more chip? How many do we have altogether now?
S: Eleven.
I: Eleven. And how about one more. How many do we have now?
S: Twelve.
I: How come you did that so fast?
S: There's eleven, then I count twelve.
I: You count 12. O.K. How about getting 2 more?
S: [quickly] 14.
I: Fourteen. O.K. How are you doing that so fast? What are you doing in your head? Are you doing something, saying some numbers to yourself?
S: I say 13, 14, like that.

We see then that Stacy could enumerate sets; could remember from one situation to the next; could add by counting on when real objects are involved and when the numbers are small. Later in the interview she demonstrated an ability to work with larger numbers. She was able to add 10 and 12 chips.

Next the interviewer wanted to see if Stacy could do addition in the absence of real objects. The interviewer took 4 chips, one at a time, and placed them behind a screen. She did the same with another 3 chips. She identified the number in each set. All Stacy could see was each chip going behind the screen. How many altogether? Stacy answered correctly.

Then the interviewer presented Stacy with 4 and 5 chips in the same manner. Again, she answered correctly. How did she do it? Previously she had denied counting on fingers. Probably the denial was the result of her teacher's strong opposition to such methods. Now, however, Stacy admitted to solving the problem by counting. She was able to do 6 + 5 (after an initial error), 10 + 4, and 14 + 6. She seemed to count on her fingers, sometimes starting from 1, and sometimes counting on from the larger number. Apparently, Stacy could solve problems involving absent objects, at least when she had the opportunity to see them, however fleetingly, before they were hidden. Is this because seeing the objects helps her to form the relevant imagery to use when they are absent?

What can we conclude about Stacy from the evidence presented so far? Initially, she did quite poorly; at the outset, her behavior seemed chaotic and she seemed retarded. There are several possible explanations of her initial difficulty. Perhaps it was largely emotional in character: She may have been intimidated by the interviewer. It is also possible, however, that the difficulty was intellectual: Stacy may have had difficulty in dealing with story material (8 cats, etc.) in the complete absence of real objects. Of course, it may well be that a combination of emotional and intellectual factors contributed to her problem; indeed, I suspect that this last hypothesis is most probable. Whatever its source, Stacy's difficulty was real and pervasive. Stacy's teacher felt that she had perhaps the most difficulty of anyone in the class, that she lacked even basic concepts like one-to-one correspondence, and that she needed the most help. Both the teacher and the interviewer (at least after the first session) concurred in seeing Stacy as severly deficient in mathematical ability, whatever the cause of the deficiency may have been.

But soon the clinical interview began to reveal some of Stacy's strengths. Perhaps this process was facilitated by the interviewer's encouragement of Stacy's counting on the fingers, which her teacher had discouraged. In any event, Stacy showed that she could perform addition by combining and counting or by counting on, when real objects were involved. She could also deal with absent objects when she was given some concrete supports. As her intellectual strengths emerged, Stacy lost much of the diffidence that characterized her earlier work: She became more assertive, and stopped shrugging her shoulders and saying, "I don't know."

This case study teaches us several things.

1. Our notion of informal knowledge--particularly counting procedures--is again extremely useful in interpreting children's mathematics.

2. A focus on the child's intellectual assets--especially his informal knowledge--helps to direct the remediation effort. The case study approach identifies strengths and suggests areas where instruction might be effective and therefore where emotional difficulties might be alleviated. It seems clear that Stacy needed a good deal of work with counting--including

finger counting--before she could to on to more formal procedures. Further, about the last thing she needed was the kind of set theoretic verbalisms encountered in her textbook.

3. The clinical interview method can be effective in making contact with children who are difficult to reach by other means. Thus, the interview was effective in overcoming Stacy's initial shyness and anxiety and in demonstrating that there was a good deal she could do.

Conclusions

Theory

The theory we have proposed seems to have some utility. It seems to give a useful analysis of the complexity of children's mathematical work, and seems to provide insight into problems of learning difficulty. One measure of the theory's value is its ability to suggest practical remediation efforts for dealing with learning difficulty. The case study method seems to confirm the theory's main principles and to suggest interesting directions for elaboration of the theory.

Case Study Method

The clinical cognitive case study method appears to be a useful technique for the study of learning difficulties and mathematical thinking generally. The method seems successful in its efforts to focus directly on intellectual processes involved in academic work and to discover new phenomena for further investigation. The method seems useful in establishing contact with children who are difficult to reach by other means. While standard tests often provide an incorrect view of children's competence, the clinical approach may be more accurate in this respect.

The clinical procedure is based on sound theoretical principles. Also, it is a subtle investigatory activity, involving the use of quasi-experimental techinques in a hypothesis-testing procedure. Given the general success of clinical techniques in psychology--e.g., the work of both Freud and Piaget--one must take them quite seriously. The clinical case study appears to be a viable procedure for the study of mathematical thinking.

At the same time, there exist many unanswered questions with respect to the clinical approach. We require investigations of such issues as the reliability of the technique and the extent to which interviewer expectancies can bias the results. It is necessary, however, to keep these possible difficulties in perspective: Standardized tests may suffer from more severe difficiencies--e.g., their tendency to misrepresent children's competence.

Implications For Research

The main implication of our work for research in mathematics education is that we require a greater emphasis on the flexible observation of children's mathematical thinking. We need as direct a view as possible of how children solve or fail to solve mathematical problems. We require techniques which permit the unexpected to happen and let us see it. If we look closely and directly at how children do mathematics, we will often be surprised at what we see. We believe that the clinical cognitive study can help to clarify our perception.

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On a Model for Teaching Young Children Mathematics[1]

Leslie P. Steffe

University of Georgia

The basic purpose of this paper is to present a model that may be useful in teaching mathematical concepts. The relationship of the model to teaching mathematics is analogous to the relationship of a blueprint to building a house. The principles that architects of blueprints use generally are taken from mathematics and from the sciences. The architect utilizes these principles in juxtaposition or in synthesis to formulate a plan, and the builder uses the blueprint to guide him in the construction of the house. But, the blueprint in no way guarantees the quality of the builder's work. In a similar way, psychological principles are used in the construction of the model presented for teaching mathematical concepts. The model, however, in no way guarantees the quality of the learning of the children, for that is largely influenced by the quality of the work of the teacher and how well the model is interpreted. To aid in the interpretation of the model, the three mathematical concepts of relation, class, and number (both cardinal and ordinal) are discussed prior to the presentation of the model, and known ways which these concepts develop in children are also presented. After these two tasks are completed, the model for teaching mathematical concepts is presented.

But before launching into the elaboration of the model, a comment concerning model-building for mathematical instruction is in order. A model of mathematical instruction can be useful as new information about the mathematical instruction may be obtained. Ultimately, however, the model must be tested in real learning or instructional settings in order that (a) basic assumptions of the model may be tested, (b) the utility of the model determined, and (c) aspects of mathematical instruction to which the model is not applicable clarified.

The model presented is a cognitive model based on developmental principles elaborated in the paper by Charles D. Smock in this collection. It is an attempt to translate those principles into principles of mathematical instruction. No claim is made that the instructional model has been shown to be valid on an empirical basis.

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[1] An earlier version of this paper appeared in a teachers strategy manual written for the Georgia Follow Through Program, Charles D. Smock, Director.

Teachers of mathematics often to not perceive the potential of cognitive development theory for teaching mathematics. But if all a child learned about mathematics had to be taught through school instruction, education in mathematics would be forced to be much more efficient than it now is. It is a mistake to assume that children begin school with little or no mathematical knowledge and acquire such knowledge from school instruction alone. Many mathematical concepts have been shown to develop through the interaction of children with their total environment and can be considered part of the basic intelligence of children at certain stages of their intellectual growth. It is important for teachers to know the basic stages of intellectual growth of children, ways children conceive of mathematical concepts at different stages, and how children shift from one stage to another for the following reasons:

1. In many cases, what a child learns from a particular bit of instruction is influenced by the stage of intellectual growth of the child.

2. Often a child does not think about a particular topic in the same way as does an adult. An adult, not knowing this, may inhibit the child's attempt to understand by imposing thinking patterns on the child in a highly symbolic form.

3. Because stages of intellectual growth are characterized by using mathematical-like concepts, an adult is able to gain insight into the thinking of the child in mathematical situations by understanding stages of intellectual growth.

4. Insight can be gained in teaching mathematical concepts from knowing how children shift from one stage to another.

The stage of preoperational representation begins around 18 months of age and lasts until around six to seven years of age. Obviously, dramatic changes take place in children during this time--the most dramatic perhaps is language development. The upper end (4-6 years of age) of the stage is of concern for mathematics teaching in preschool or early elementary school. The stage of concrete operations begins around six to seven years of age and lasts until about 11-12 years of age. It must be emphasized that the age break between the two stages varies from child to child. One cannot expect age to determine exactly the stage of the child. Some children do not reach the stage of concrete operations until after eight years of age, whereas some children may reach the stage as early as five years of age. Every child goes through the stages in the same order but not necessarily at the same rate. In the following material, the stages of intellectual growth are presented by selecting particular mathematical concepts and describing known ways that children deal with the concepts.

Relations

Children's Conception of One-to-One Correspondence

Children in the stage of preoperational representation do not conceive of the relations in the same way as do older children. Some examples are given of how children at the stage of preoperational representation have difficulty.

Example 1: Establishing a correspondence. Imagine that two children drop beans into one jar (Figure 1) in such a way that for each lima bean one child drops, the other drops a navy bean. Suppose the lima beans are exhausted before the navy beans. From the action of placing beans into the jars, the children should know that (a) each child put as many beans in the jar as the other, (b) the child who has navy beans remaining has more than the other, and (c) the child who has lima beans has fewer beans than the other. The children should also realize that the navy beans consist of the remaining navy beans as well as the navy beans in the jar. Five- and six-year-old children (especially the five-year-olds) have difficulty interrelating such knowledge. They may think that the child with navy beans has more beans in the jar than the other child because they may not distinguish the navy beans in the jar from those remaining and make the comparisons of the beans in the jar based on those remaining.

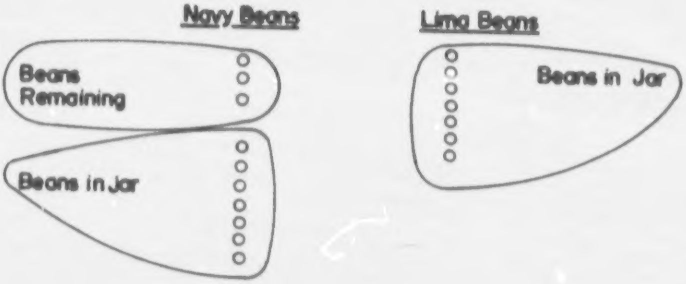

Figure 1

This example is important in cases where children are being taught to order the whole numbers. At least two distinct varieties of tasks are used in establishing, say, that eight is less than nine and that nine is greater than eight. On the one hand, two distinct collections of objects, one of eight and one of nine, are presented to the child who is required to match the objects of the two collections one-to-one. Based on this matching, the child is expected to realize that eight objects match one-to-one to a subcollection of the nine objects; that there are more objects in a collection of nine objects than in any one of its subcollections; and finally, based on these two reasons, nine objects are more than eight objects. Except for context, this ordering task is analogous to the one presented in the previous example. Children at the stage of preoperational representation have difficulty in conceiving of a total collection and two of its subcollections at the same time. If they think of the total collection, they may lose sight of the subcollections, and if they think of the subcollections, they may lose sight of the total collection and make the errors described in the example of placing beans in the jar.

On the other hand, if one collection of nine objects is presented to a child who is required to count them, that nine is more than eight is established through noting that eight is the number of objects in a subcollection of nine objects. Here again, the child is expected to realize that any subcollection of a collection of objects has fewer objects than does the original collection.

Example 2: Conserving the relation. To say that a child should be able to determine the correct matching relations between two collections of objects would be artificial if the relation, once it is determined, could not be conserved by the child. Consider the case where a child constructs a matching as pictured in Figure 2 and says that there are more circles than stars. Then if the stars are altered in full view of the child to the display, as in Figure 3, and the child thinks that there are more stars than circles, it would not be correct to say that he can determine a matching relation between two collections of objects except in a most superficial manner.

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Figure 2

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oooooo

Figure 3

Example 3: Effects of perception. Imagine an experimental setting where two transparent glasses are present of the same size and shape with two piles of beads: blue beads and red beads. The child and the experimenter place beads in different glasses at the same time--the experimenter blue beads and the child red beads--one after the other until the two glasses are full and contain the same number of beads. The child says, "They're both the same." When asked how he knows, the child replies, "Because they are the same." The experimenter and child then take the beads, bead by bead, simultaneously from the two glasses and place them into two other glasses, one taller but narrower than the first glasses and one shorter but wider than the first glasses. After completion, the child is asked whether there are now the same amount in both. He responds, "No, here (the taller glass) there are more because it's big."

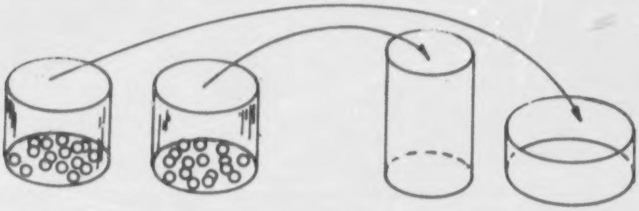

Figure 4

These responses are characteristic of the stage of preoperational representation. The perceptual features of the situation completely override the knowledge gained through the correspondence, and the correspondence is not lasting in the face of the beads having differing shapes in the two final containers. The relative heights of the blue beads and red beads serve as the basis for the final judgement that there were more blue beads than red beads. One could argue that no correspondence existed for the child when the beads were in the identically shaped containers--that the initial judgement of equality of the beads was based on the shape and size of the two glasses. The action of placing beads simultaneously into two glasses did not result in a quantifying correspondence for the child. In the experiment, then, the quantitative judgements made by the children could be categorized as gross quantitative judgements.

Example 4: Correspondence in "transitional" children. Imagine an experimental setting where the child is asked to select the same number of candies as there are in a row of seven candies. Various types of responses are possible on the part of the child in the stage of preoperational representation. Quite often such children will make a row of candies the same length as the given row but ignore the number of candies in the row they construct. The placement of the correct number of candies in their row is purely an accident. These children do not coordinate the length of the row of their candies with the density of the candies in the row. They focus on only one feature (length) of the situation at a time. It would be a mistake to say these children have a conception of one-to-one correspondence or even attempt to construct one. They react purely on the perceptual features of the configuration of the candies without coordination of those features.

Another level of responses has been identified which goes beyond those above. Children may coordinate the length and density of the two rows of candies, making two rows of equal length and density, but not conserve the one-to-one correspondence if one of the rows is spaced close together or further apart. Such children do believe that if the two rows of candies are identically spaced there will be the same number in each row. Consider the child who made a row of six candies correspond to a given row of six (the model row) by spacing them equally with the model row, but when his row was crowded together, he then thought there were more in the model row. When asked to make the two rows have the same number of candies again, he spaced his row identical to the model row, not adding nor taking any candies away from either row. This child was definitely more advanced than the children at the stage of preoperational representation (who are only capable of gross quantitative comparison), but does not fully comprehend the concept of one-to-one correspondence. His concept is transitional from essentially no concept to an operational concept of one-to-one correspondence. This child's concept of one-to-one is said to be in a transitional stage.

Example 5: Operational one-to-one correspondence. Imagine a situation of five toy cars corresponding to five toy garages, the red car in the red garage, the blue car in the blue garage, etc. The one-to-one correspondence is based on the color of the cars and garages. It is qualitative in nature because it is based on the qualities of the elements. But does it go beyond a qualitative one-to-one correspondence? It could be an intuitive or an operational one-to-one correspondence, intuitive if it is not conserved and operational if it is. Being operational means the elements are considered as units--that is, any car can be placed in any garage--the color is irrelevant to the fact that there are the same number of cars as garages. A given car can be considered to be a placeholder for any other car. They can be exchanged without a loss in the one-to-one correspondence.

Example 6: Transitivity and one-to-one correspondence. Knowing that three levels of one-to-one correspondence exist for children is important in planning learning activities for children involving one-to-one correspondence. The three levels are (a) no one-to-one correspondence, (b) intuitive one-to-one correspondence, and (c) operational one-to-one correspondence.

Consider the following problem. Twelve buttons are on a table in front of a cardboard box from which the top and front are removed. A partition divides the box into halves; ten checkers are attached to the bottom of the box on one side of the partition, and ten tiles are attached to the bottom of the box on the other side of the partition. The child is asked to find out if there are as many checkers as tiles by using the buttons. Following are protocols of two children who tried the problem:

Deb: (5 years, 11 months).

Experimenter: Deb, find out if there's as many checkers here as tiles here. Use these buttons to find out. (Deb pairs buttons and tiles.)

Deb: Ain't no buttons over there.

Experimenter: No. We just have one pile of buttons.

Deb: Yellow ones is my favorite color--I got a good idea I can do.

Experimenter: What is that?

Deb: Wait a minute--put two on each one of 'em.

Experimenter: What are you doing now?

Deb: I had to pick up the two lemon ones.

Experimenter: The two lemon ones?

Deb: Yea.

Experimenter: What are you doing with the buttons now?

Deb: Pairing them.

Experimenter: Pairing them with what?

Deb: With the tiles and the checkers.

Experimenter: Are there as many checkers as tiles?

Deb: No.

Experimenter: How can you tell that?

Deb: Cause these two, these two, these two, these two are missing. Yes.

Experimenter: Yes, there is?

Deb: Yes.

Experimenter: Well now, when you had all the buttons over here except the two lemon ones, were there as many buttons in there as checkers?

Deb: No.

Experimenter: But when you had the two lemon ones here and had all the other buttons in there with the checkers, were there as many buttons as checkers?

Deb: Yes. No.

Experimenter: Let's put them back to see.

Deb: Oh, yes.

Experimenter: All right. Suppose you hand me the two lemon ones, and suppose I keep them for you. Now, I wonder if there's as many buttons as tiles?

Deb: Let me see. See my new shoes. You got your Easter shoes?

Experimenter: No, I don't have my Easter shoes.

Deb: I don't have mine either. It's as many.

Experimenter: There are as many buttons as tiles? (Nod, indicating yes.) Okay, are there as many checkers as tiles?

Deb: I don't know because I can't pair the checkers and the tiles together.

Experimenter: You can't?

Experimenter: Did you have as many buttons as tiles?

Deb: Yes.

Experimenter: Okay, how about the checkers and tiles?

Deb: I don't know. (Then Deb points one finger at a checker while pointing another finger at a tile. She then announces:) Yes.

Experimenter: How do you know?

Deb: Cause I point my finger at each of 'em.

Experimenter: I see.

Tom: (Six years, 0 months).

Experimenter: What are these objects, Tom?

Tom: Checkers.

Experimenter: These.

Tom: Tiles.

Experimenter: Tom, I want you to find out if there are as many checkers as tiles. You may use the buttons to find out if there are as many checkers as tiles. Okay, how can you use the buttons?

Tom: Pair 'em--

Experimenter: Tom, tell me what you have done.

Tom: Paired the checkers with the buttons.

Experimenter: What did you find out?

Tom: As many buttons as checkers.

Experimenter: But I want to find out if there are as many checkers as tiles. Can you do that using the buttons? (Tom pairs buttons and checkers.) Tell me what you have now.

Tom: As many tiles as checkers.

Experimenter: All right, how do you know that?

Tom: Cause both of these buttons don't have a checker and a tile for a partner.

Experimenter: All right. There were two buttons left here when we paired them with the checkers?

Tom: Yes.

Experimenter: And there are two buttons left now when they're paired with the tiles?

Tom: Yes.

Experimenter: Does that make it as many tiles as checkers?

Tom: Yes.

Experimenter: Thank you.

Deb never solved the problem of comparing the checkers and tiles. Her comment, "I DON'T KNOW BECAUSE I CAN'T PAIR THE CHECKERS AND TILES TOGETHER." is most revealing--she can only find out through direct comparison. Her concept of one-to-one correspondence is not operational. The correspondence she established between the buttons and checkers has little significance for her when comparing the two static collections (checkers and tiles). It was as if the first correspondence between the buttons and tiles never existed. Tom's solution was sophisticated in that he used the two remaining buttons for his comparison. For him, the one-to-one correspondence established in both cases was related and quantified the sets. One-to-one correspondence for Tom was operational. He was able to use one-to-one correspondence in problem solving, whereas Deb was not able to do so.

In summary, children's conception of one-to-one correspondence passes through three stages: no one-to-one correspondence, intuitive one-to-one correspondence, and operational one-to-one correspondence. In the two first stages, children do not conserve one-to-one correspondence. However, intuitive one-to-one correspondence is definitely an improvement over no one-to-one correspondence. Children in the stage of preoperational representation are in the first stage of one-to-one correspondence, children in the transitional stage are in the intuitive stage of one-to-one correspondences, and children in the concrete operational stage are in the operational stage of one-to-one correspondence. Children in the operational stage of one-to-one correspondence are able to use one-to-one correspondence in solving problems involving the principle of transitivity (example 6) and the notion of an arithmetical unit (examples 4 and 5). The concept of one-to-one correspondence is quite well developed for these children.

Classification and Equivalence Relations

If a child is given a collection of sticks and asked to put the sticks into piles so that the sticks in any one pile are the same length, he must base his classification activities on the relation "the same length as." It would seem essential for the child to employ properties (reflexive, symmetric, and transitive) of this equivalence relation. In order to clarify this, the following analysis is given of a child's behavior in sorting a collection of sticks into piles.

The child must select a stick (say s) from a given collection of sticks and search for another stick r the same length as s. If no such stick r exists, then the child must classify s with itself (reflexive property). If some stick r does exist the same length as s, the child must realize that not only is s the same length as r, but r is also the same length as s (symmetric property) for r and s to be considered as forming a class. Given that r and s are classified together, then the child at some time must hunt through the sticks yet to be classified to determine if there is another stick the same length as s (and r). Suppose that such a stick t exists. For t to be classified with r and s, the child must realize that all three are the same length, which entails knowledge of transitivity of "same length as" (s is the same length as t and r is the same length as s, so r and t are the same length). One may object and say that all the child has to do is compare t and r and s by placing the sticks together, so transitivity has never to be used! Our contention is that children would not think of classifying the sticks together in absence of the ability to employ transitivity, the reflexive property, and the symmetric property.

To obtain an idea of what classification abilities to expect from first and second grade children, 81 first and second grade children (39 first and 42 second) were given three tasks to perform. In the first task, the children were given a collection of sticks to sort into three collections and three sticks on which to base the sorting. Each of the sticks in the collection was exactly as long as one of the three sticks. Sixty-two of the children correctly sorted the sticks into three piles.

In the second task, the children were given another collection of sticks to sort but were not given three sticks on which to base the sorting. The children were asked to put together all of the sticks that belonged together. Six of the 81 children did not attempt the task. However, only 37 children completed the second task as compared to 62 in the first task. The 44 children who did not complete the task made some piles, but did not mentally connect together piles that went together and generally placed sticks into piles incorrectly.

The third task consisted of giving the children three piles of sticks already sorted together on the basis of "same length as" and asking them why the sticks were put together in the way they were. Fifty-nine children did not discover the basis for the classification (same length as). The remaining 22 children showed some evidence of being aware of the classification.

Classification has been studied more widely than just in relation to "the same length as." In order to fully appreciate classification behavior of children, it is necessary to discuss classes (or sets) per se. Generally, when objects are classified together, they share common properties. For example, quite dissimilar objects can be classified together under the heading "fruit." What makes these objects "fruit" is what is common. Within the class of fruit, however, important differences exist--oranges and apples are different. Given a universe of objects, three distinct kinds of properties exist (Inhelder & Piaget, 1969).

1. Properties specific to members of a given class (e.g., the properties which make items fruit) which distinguishes the class from other classes (from vegetables, meat, etc.).

2. Properties which are common to members of a given class and those of other classes to which it belongs (e.g., that which is common to fruit and vegetables).

3. Properties which differentiate members of a given class one from another (those which differentiate a pear from an apple, for example).

Part-whole relations of class membership and inclusion also exist. These relations are conveyed by the terms "all," "some," "one," "none," when applied to the members of a given class and those of the classes to which it belongs (all oranges are fruit, some fruit are apples, no fruit are vegetables).

The intension of a class is the properties common to the elements, and the extension of a class is just the members of the class. The coordination of the intension and the extension of a class is what develops in children in stages. These stages correspond to the three stages identified in the development of one-to-one correspondence.

Young children below about six years of age employ primitive behavior in attempting to form classifications. The types of collections formed by these children have been called complexive collections or graphic collections. For example, children were asked to classify a collection of geometric objects together, some triangular shapes, some square shapes, and some half ring shapes. At least three varieties of graphic collections were identified. First, some children constructed a number of subcollections, ignoring the rest of the material which was never classified. The subcollections had no common property--the child would change criteria of classification within a subcollection. Some times, subcollections were not formed but properties of individual items noted.

Second, successive similarities between one object and the next were formed. While this is an improvement over the type of behavior noted in the first example, it is not true classification as no overall criteria for classification was found for subcollections; subcollections were not differentiated, and part-whole relationships were not identified.

Third, definite figures are made out of the objects--a "house" is made, then windows, etc. That is, the child makes no real attempt at classification, but instead plays with the objects, making whatever comes to his fancy.

The graphic collections described above have two features differentiating them from true classes. First, [illegible] collections are formed on the basis of the spatial arrangement of the objects. Second, no criteria for classification (no properties which [illegible] all the elements together) were isolated by the children. These two aspects are simply another way of saying that intensive properties were not identified by the children--these children are at Stage I (preoperational) as regards their classification behavior.

Stage II (or transitional) classification behavior is an advance over Stage I classification behavior, but it is not yet operational classification behavior. Stage II classification behavior can best be characterized by a recognition of intensive properties, with no complete coordination between the intension of a class and the extension of a class. Given a class of objects, children are able to separate the class of objects into subclasses. This means that they understand that all elements can be classified, each subclass contains elements of a specific kind or which possess a specific property, and two or more subclasses are constructed. Yet, the subclasses formed are not thought of as forming a hierarchy of classes. The class-inclusion relation is not mastered.

The class-inclusion relation being mastered means simply that, given a class A which is contained in a class B, the child understands all of the A are some of the B but all of the A do not constitute all of the B. For example, if A is the class of Siamese cats and B is the class of cats, then all Siamese cats are certainly cats, but they do not exhaust the cats. That is, there are cats that are not Siamese cats. So, all of the A do not constitute all of the B, but just some of the B. Children at the transitional stage of classification certainly realize that Siamese cats are indeed cats and, in fact, are part of the set of cats. So, one would think they would understand class-inclusion. But they do not. It is critical they understand that there are other cats than Siamese cats or, in other words, that all cats are not Siamese. If A' are the non-Siamese cats, then $A \cup A' = B$ and $A = B - A'$ (see Figure 5).

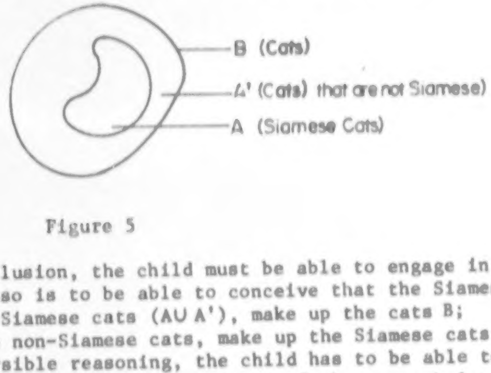

Figure 5

To understand class-inclusion, the child must be able to engage in reversible thinking. To do so is to be able to conceive that the Siamese cats, together with the non-Siamese cats ($A \cup A'$), make up the cats B; and that the cats, minus the non-Siamese cats, make up the Siamese cats ($A = B - A'$). In this reversible reasoning, the child has to be able to conceive of the total class of cats as being made up of the two subclasses at one and the same time. Stage II children, when focusing on the cats, lose sight of the subclasses, and when focusing on the subclasses, lose sight of the total collection. Typical responses of transitional children (Stage II) are given in the following situations. A picture is shown to the children on which there are, say, four Siamese cats and three cats which are not Siamese. The children are asked to compare the number of cats to the number of Siamese cats. When asked to do so, the children will compare the Siamese cats to the other cats.

The children at Stage III (concrete operational) are capable of solving the class-inclusion problem and are much more flexible in their classification behavior than are Stage II children. Stage II children are able to build hierarchies of classes. For example, they are capable of conceptualizing such hierarchies as Maltese terriers are part of the terriers, terriers are part of the dogs, dogs are part of the mammals, etc. Stage III children are not only capable of building hierarchies of classes, but are able to change the criteria of classification and reclassify a set of elements in a new way. The child may consider new dogs in his classification and refine the classification to include many more classifications than those given. Two complementary processes exist that describe the Stage III flexibility in classification. One, given a classification, the child can go back and construct finer classifications or whole new classifications and not be tied to the one constructed. Two, a child can anticipate a classification before it is done.

In summary, the following three stages in children's classificatory behavior have been identified (Inhelder & Piaget, 1969):

> Stage One. (Preoperational) Given a collection of objects and told to "put everything together that goes together," a child at this stage forms what is known as "graphic collections." If he does anything, he constructs one or more spatial wholes. This is a child's first attempt to coordinate part-whole relations with those of equivalence and difference.
>
> Stage Two. (Transitional) At this stage, the constructed collections are no longer graphic collections. Trial and error plays a large role in construction of classifications and no overall plan is present. Children cannot yet solve the class-inclusion problem but do understand that all elements need classifying, each subclass contains elements which possess a specific property, and two or more subclasses are constructed.
>
> Stage Three. (Concrete Operational) Children at this stage are able to coordinate the intension and extension of a class, as evidenced by the solution of the class-inclusion problem. Children at this stage are capable of conceiving of hierarchical arrangements of classes, and are capable of imposing more than one classificational system on the same collection of elements, anticipating the new classification systems before carrying out the classification.

Order Relations and Seriation

Order relations determine a seriation of the objects on which they are defined just as equivalence relations determine a classification of the objects on which they are defined. Three stages exist in the development of seriation behavior (Inhelder & Piaget, 1969).

> Stage I. (Preoperational) This stage is characterized by no attempt at seriation or the forming of small uncoordinated series.

Stage II. (Transitional) This stage is characterized by seriation by trial and error.

Stage III. (Concrete operational) This stage is characterized by a systematic method of seriation.

At Stage I, the child either does not attempt to form a series or else forms small uncoordinated series of two or more elements. In the latter case, the subseries are not connected by the child (ıl ıl ıll). The representation in parentheses is supposed to connote that the child first orders two sticks, then two more, then three more, never realizing that the sticks need to be ordered into one series.

At Stage II, the child is not systematic. He can form a series, but does so with no overall plan nor complete anticipation of what he is to do. For example, a child may pick two sticks and put them in order, pick two more and then put them in order, and then attempt to coordinate the four sticks into an order, etc. Or, a child may lay a whole "series" out and then attempt to put them in order through a process of trial and error. This is an advance over Stage I seriation behavior.

At Stage III, children proceed systematically, e.g., choosing the smallest element (or largest, depending on where they start), then the next smallest, etcetera, until they are done. These children know beforehand that a given stick (say the third) is going to be longer than those already chosen, but shorter than all those yet to be chosen. Children at the second stage do not realize this, being capable of thinking in one direction only.

Children's Conception of Number

Classes (sets) and relations logically are fundamental to number, both cardinal and ordinal. Because of this logical relationship among classes, relations, and number, the material on children's conception of classes and relations is pertinent to the discussion on children's conception of number.

One-to-one correspondence is essential to class usage of cardinal and ordinal number. In the section entitled "Children's Conception of One-To-One Correspondence," three types of one-to-one correspondence were identified from the point of view of the child, no one-to-one correspondence, intuitive one-to-one correspondence, and operational one-to-one correspondence. These three types of one-to-one correspondence determine the quantitative judgments of which children are capable--gross quantitative judgments, intensive quantitative judgments, and extensive quantitative judgments. The gross quantitative judgments are based on the perceptual features of the situation--but only one at a time. For example, if two rows of several candies are arranged so one is longer than the other (State 2 in Figure 6) the child capable of gross quantitative judgments may say there are more in the longer row, not paying attention to the density of the candies in the two rows, even if they had been previously arranged (State 1 in Figure 6) so as to be the same length and density. In the case of State 1, the child may have said that the two rows of candies had the same number, but yet, not recognize the contradiction in his judgments about the two states.

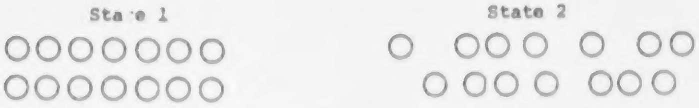

Figure 6

For the judgments were based on the apparent reality--the perceptual configurations. Essentially, no one-to-one correspondence exists for the child.

The child capable of intensive quantitative judgments would begin to coordinate the length and density of the objects in the two rows in State 2, but would not yet realize that the increase in length in row 1 is exactly compensated for by a decrease in the density of the objects. He would only admit that if the two rows are put back as they were in State 1, that they would be of the same length.

The child capable of extensive quantitative comparisons would immediately say that the two rows in State 2 are of the same number because he views the objects as units. For this child, equal number and greater length implies less density. Here, the one-to-one correspondence is operational for the child.

As noted earlier, if the one-to-one correspondence is operational for the child, he should be capable of engaging in transitive reasoning--the notion of equivalent sets becomes operational. The child, of course, does not know the symbolism involved, nor is he aware in any way of set theory. But he is able to reason in concrete situations involving collections of objects.

The child, then, for whom one-to-one correspondence is not operational, would not be capable of the class usage of number, either cardinal or ordinal. He may know number names, however, and be able to associate them with specific collections. On the other hand, the child for whom one-to-one correspondence is operational should be capable of the class usage of number, both cardinal and ordinal. The class usage of cardinal and ordinal number involves classification, where the classification is based on one-to-one correspondence (set equivalence). In fact, if a cardinal number, such as 4, is viewed as a particular set, such as {a, b, c, d}, then surely classification is involved even in the member-of-a class meaning of cardinal number. If {a, b, c, d} is considered as an ordered set, then an asymmetrical transitive order relation "precedes" is involved in the member-of-a class usage or ordinal number as well as classification. So, order relations and, hence, seriation is involved in the notion of ordinal number, and classification is involved in both cardinal and ordinal number even in the member-of-a-class usage.

It should be clear that if a child is at Stage 1 (preoperational) in classification, seriation, or one-to-one correspondence, he doesn't have much chance of dealing with cardinal or ordinal number on any except the most superficial of levels. It must be emphasized that there is nothing "wrong" with a child who is at Stage I on any of the above. All children pass through the stages identified. At Stage II (transitional) in classification, seriation, or one-to-one correspondence, children are beginning to deal with conceptual aspects and definitely are progressing to a more advanced stage of dealing with cardinal and ordinal number. Stage II classification behavior was characterized by children recognizing the intension of a class, but yet, with no coordination between the intension and extension. Children at Stage II are able to partition a class of objects into subclasses, but the subclasses formed are not thought of as forming a hierarchy of classes--the class inclusion relation is not mastered, and hierarchical classification systems are not conceived of. Consequently, there is a good possibility that children at Stage II classification behavior are able to deal with the member-of-a class meaning of cardinal number in terms of relatively small numbers of elements (less than seven). However, even though a Stage II child is able to cite verbal number names in order, one should not take that to mean that the child is forming successors of sets--or a sequence of ordinal numbers. His number names constitute a verbal chain, each individual number having a referent, but the numbers are not "nested" for the child because he is not yet capable of forming a hierarchy of classes, which is necessary in dealing with ordinal numbers, and hence, counting.

Stage II classification behavior emerges at about the same time as Stage II seriation behavior, so that the beginning of the member-of-a class usages of ordinal number is beginning for the child. However, it is not until Stage III that the child comes to a conception of number in its operational sense. He is now capable of classification and seriation and conceives of equivalence and order relations in the sense that thinking follows these relational patterns.

A Model for Teaching Mathematics

It is now time to turn our attention to possible relationships between school learning in mathematics and stages of intellectual development. The information presented to this point might suggest that one should not present the topics of sets, relations, and number to children in the stage of preoperational representation. The situation, however, is not as clearcut as that. The information that has been presented is based on what is known about how children reason when that reasoning involves equivalence or order relations, sets, and number, not on the way in which that reasoning develops. The following discussion on factors contributing to development will shed more light on how such reasoning develops, and how one could influence that development (Piaget, 19

Maturation

At least four factors have been identified which contribute to the development of cognitive growth of children--and specifically, to the development of mathematics in the child. The first is maturation. In support of the proposition that maturation is involved in cognitive growth, it is a fact that transitive reasoning has seldom been observed in children four years of age or younger. While that statement cannot be taken as proof that maturation is involved in development, it certainly indicates that maturation does play a prominent role. If it played no role in development, then subjecting a child to learning experiences would be sufficient for him to gain an understanding of the concept or principle involved. Evidence does exist, however, that great difficulty in learning transitivity of "as many as" exists for children in the stage of preoperational representation, even when apparently appropriate learning experiences have been encountered. Thus, it seems likely that the changes occurring as the child grows older make possible learning which was not previously possible.

Experience

Experience by itself does not explain conceptual growth of children but experience does play an important role in conceptual growth. Too much variation has been observed in the age of attainment of the stage of concrete operations to discount the role of experience. But experience alone does not explain the growth of mathematical concepts. Otherwise, as already noted, all one would have to do to "teach" any child transitivity would be to give him sufficient experience--and he would learn. But, unfortunately, it is not that simple.

Physical experience and mathematical experience. Experience should be analyzed in two ways. One is in terms of physical experience, and the other is mathematical experience. To make a distinction between these two types of experience, imagine a child matching the objects of set A one-to-one with the objects of set B through overt actions. He places one object from set A with one object from set B, etc., until all the objects of one or both of the sets are exhausted. Then he takes the objects of set B and likewise matches them with the objects of another set C. Now, does this matching constitute a physical experience or a mathematical experience? The answer is that it could be either one, depending on the child. One cannot differentiate between the two types of experiences through observation of the overt acts of matching in which the child engages. The crucial determiner of the type of experience is whether the sets A and C are related by the child by virtue of the comparisons of A and B and B and C. If the child is not able, through reasoning, to determine the relation between A and C, then the experience gained through overt matching of the objects of A and B and B and C was mainly physical in nature. The relation between the sets A and B, in this case, was a function of the physical arrangement of the objects and would not exist for the child in the absence of the physical pairing of the objects. The relation would be external to the child and would be destroyed upon rearranging the objects of the sets. While the two sets of objects were in a state of physical comparison, the child could definitely obtain knowledge about the objects--either they match or they don't--but for the knowledge to be mathematical in nature, the relation must be conserved by the child when the objects are moved to new states, and the child must be able to engage in reasoning involving the properties of the relations.

This distinction between a physical experience and mathematical experience is important in understanding the growth of mathematical concepts. Maturation contributes the internal mechanisms to this growth that allows the child to go beyond physical experience. Development must be supported by experience--a child does not mature in a vacuum. The boundary between maturation and experience is not known--that is, their relative contributions to the cognitive growth of children are not known. However, as has been emphasized, each is important to conceptual development.

More must be said concerning the distinction between physical experience and mathematical experience. If a child is wrong, it is easy to show him he is wrong if his knowledge is from physical experience alone. Whereas, in knowledge derivable from mathematical experience, if a child is wrong, it is generally quite difficult to show him that he is wrong. Verbal transmission of the correct answer to the child is most often insufficient to show him he is wrong. For example, when overtly comparing two sticks, if a child fails to align two endpoints correctly, it is easy to correct his mistake. If, however, he fails to display transitive reasoning in a task, it is very difficult to demonstrate transitivity to him in one or two examples. In the case of transitivity of "as many as" given above, the relation between A and C has to be inferred by the child because the elements of A and C were not directly matched. There was no physical experience on which the child could rely. On the other hand, when physically comparing two sticks, the child can be shown through a physical action if he is wrong in his alignments of the endpoints. Another example is where a child has eight wooden beads, three white and five brown. If he is asked whether he has more brown beads or more wooden beads, and if the child errs, again it is quite difficult to show him he is wrong. Knowledge acquired through physical experience alone is worth knowing and often is the source of observations leading to more organized knowledge.

Knowledge derivable from physical experience alone is called physical knowledge. Physical knowledge is characterized by knowledge about the properties of objects. Physical experience is generally thought of as experience through direct contact with objects through one of the five senses. For example, one may touch something and it is hard, cold, hot, soft, supple, etc. Or, one may see something--an object is red, a diamond cutting glass, the shape of a banana, etc. Knowledge gained through these observations is concerned with the properties of the objects. An observer may bring something to the observation which allows him to go beyond physical knowledge and gain logical or mathematical knowledge about the objects not possible by another observer. An example is that while redness is a property of an object which is in fact red, a property which exists independent of the observer, light can also be described as wave motion, so that an observer who knows this may experience the redness at the level of mathematical experience. Another example is where an observer sees an iron boat floating and another piece of iron sink. These two observations can be summarized by saying, "sometimes iron floats and sometimes it sinks." This knowledge is pure physical knowledge if the implicit contradiction is not removed. Its removal demands a mental construction beyond the physical knowledge gained through direct observation. Knowledge gained through observation alone is at best fragmentary if not connected by principles.

Even though an experience which is the result of direct contact with an object through the senses may go beyond physical experience (depending on the observer), the source of mathematical experience is generally thought of as being overt actions. This does not mean that just because a child is involved in overt actions, he will be having a mathematical experience. It has been already noted that a child may overtly match the objects of two equivalent collections, and the knowledge gained from the overt matchings may be nothing more than physical knowledge. This is especially true for children at the stage of preoperational representation and those capable of only intuitive one-to-one correspondence. Those children capable of one-to-one correspondence would, by definition, gain mathematical knowledge from the overt actions. A critical difference is that the mathematical knowledge gained demands that a pair of physical objects not be defined by the closeness of the objects. Two objects may make a pair even though they are quite far apart. An example often cited of mathematical experience is where a child realizes that it makes no difference how you count a collection of objects--you get the same number. This knowledge is gained through counting in at least two ways.

Linguistic Transmission

Another factor contributing to the growth of mathematical concepts is linguistic transmission of information (information transmitted through language, oral or written). This factor includes the verbal interchange of the child with other people. As such, it is to be considered as a part of the experience of the child. However, experience goes beyond linguistic transmission, so what is said about the latter is not necessarily true of the former.

Certainly, a child can receive valuable information via language. One must not think, however, that information contained in a verbal communication will necessarily increase a child's understanding of a mathematical concept. To verify this statement, a transitivity problem was presented to 40 first graders, 40 second graders, and 40 third graders who were in the top two-thirds of their classes according to teachers' judgments. Each child was presented with a transitivity problem involving length relations. If they could not solve the problem, they were told the correct relations which held between the two sticks. For example, if a child took a stick B and compared it with C and then with A and found that A and B were the same length and B and C were the same length, then could not infer the relation between A and C, they were told that A and C were the same length. After being told, the child was asked to explain why A was as long as C. Of 24 children who could not infer the correct relation between A and C, only five could explain why A and C were of the same length after being told. Even though the children were not told the reason A and C were the same length, they had just gone through the two comparisons and said that A and B and B and C were of the same length. Being told that A and C were of the same length was not sufficient for 19 of the children to go back mentally over their actions and gain information from reflecting on them. The actions became significant for only five of the children.

The above example illustrates the point being made. Attempting to teach the preoperational child mathematics by only verbal or symbolic means has the potential of leading to disasterous results. But, because words and symbols are a part of mathematics teaching, their role must be further clarified. Experience has taught us that there should be a continual interplay between the spoken words which symbolize a mathematical concept and the set of actions a child performs while constructing something that makes the concept tangible. In short, there is good reason to develop mathematical vocabulary during the course of activities used to develop the concept. The particular blend has to be determined by the specific activity and child engaging in the activity. But it is a long way from promoting vocabulary development to recommending that the teaching of mathematics to young children be based on symbols of mathematics or verbalization.

Equilibration

The last factor important in the growth of mathematical concepts is a principle called equilibration. Of the four factors wich contribute to the growth of mathematical concepts, this factor is the most fundamental but the most difficult to employ in practice. There is not much one can do to vary maturation, short of being sure the child is physically healthy. So, while one must acknowledge and understand maturation, essentially there is little a teacher of school mathematics can do to control it. Experience and linguistic transmission are under control of the mathematics teacher, but only in so far as the "mathematical" experiences of the child are concerned. Here it is crucial to distinguish the different levels of experience and appreciate the role of language in planning the mathematical environment for the child.

But little has been said yet about knowledge acquisition--that is, is there anything which would help to understand how a child acquires mathematical knowledge? In the case of relations, either children have little or no knowledge of relations, they are able to engage in reasoning involving the properties of the relations, or they are oscillators. If they are able to reason involving relations, that reasoning is limited and can be extinguished quite easily. The difference between physical experience and mathematical experience helps to clarify the role of manipulative activities in the classroom. Some children may engage in a manipulative activity but yet be involved only at the level of physical experience, whereas another child may be involved at the level of mathematical experience. But the question of how to maximize the possibility of a child engaging in the activity at the level of mathematical experience remains yet unanswered. Stated another way, are there any clues which one could use in taking a child from a physical experience to a mathematical experience? One of these clues is equilibration. The reader is referred to the paper by Smock in this collection for a full elaboration of the concept.

Learning-Instructional Phases for Mathematical Concepts

One basic assumption in this document is that most mathematical concepts go through levels for the person who is learning the concept. For concepts not shown to be developmental, these levels should not be confused with stages in development for concepts which have been shown to be developmental. One essential difference is that stages of development are the result of a child's interaction with his total environment and occur in every sane person. For most mathematical concepts to occur, special learning environments must be created. But the creation of the learning environments does not insure that a concept will be learned. The notion of learning-instructional phases elaborated below is useful in creating appropriate environments for concept acquisition. The first learning-instructional phase, called exploration, corresponds essentially to a first level of concept--that of no concept. The second learning-instructional phase, called abstraction and representation, corresponds to a second level of concept. The third learning-instructional phase, called formalization and interpretation, corresponds to a third level of concept.

Exploratory Phase

The first learning-instructional phase is called exploratory. Children's play is considered as a critical part of this learning-instructional phase. Play is viewed as an assimilatory activity and is an essential part of early mathematics learning. Whenever assimilation occurs, its counterpart, accommodation, also occurs if equilibration is to operate. Equilibration is viewed as one essential factor of development and is now extended to learning mathematical concepts in general.

Equilibration is a useful theoretical construct to help guide learning activities in mathematical instruction. By itself, however, it provides few hints in specific situations. More is needed in order to employ the principle in practice. Play activities, as a teaching technique, should be thought of as corresponding mainly to the first level of mathematical concepts identified --essentially no concept--and to the second level--rudiments of the concept. At these two levels of mathematical concepts, a great deal of constructive thinking needs to be done by the child. It is a period of concept formation, not analysis. More analytical thinking must come after something exists to be analyzed.

Play activities can vary along two quite important dimensions. The first is the type of material, and the second is the external direction the child is given. The materials can vary from structured to unstructured, and a play activity can vary from highly directed to undirected. In the latter case, it is important to realize that the child usually structures his own play activities.

Multiple embodiment principle. In order to illustrate the principles of self-regulation and play activities using particular concepts, imagine that a teacher decides that one-to-one correspondence is to be worked on. In this case, no concept of one-to-one correspondence corresponds exactly to the preoperational stage of development. Problem situations have been given which can be used to approximate which of her children are without a concept of one-to-one correspondence. After such an approximation is made, the teacher should allow the children who do not display any concept of one-to-one correspondence to engage in undirected play activities, using physical objects which will later be used in directed play activities. For example, the teacher may have assortments of beads, bird cutouts, blocks, discs, animal cutouts, toy animals, toy cowboys, toy soldiers, toy guns, dolls, dresses, toy dishes, or toy utensils.

Let's take a particular free play activity where the preoperational children involved place cowboys and Indians on horses. Through this assimilatory activity, the children can gain the physical knowledge that indeed the cowboys and Indians fit on the horses. The teacher can not employ an artful suggestion which may create a disequilibrium or mismatch for the children. For example, she may suggest that the children find if there are enough cowboys and Indians so each horse would have a rider. In order to find out, a child has to accomodate his practical or symbolic activity to engage in a goal directed activity. If the children do so, the teacher can employ the multiple embodiment principle (Dienes, 1971) and give them new materials with a similar goal. (For example, are there enough dresses so one could put a dress on each doll?)

If the children do not initiate their own goal directed activity upon suggestion, she may directly show them how to find out if there are enough cowboys and Indians so each horse has a rider. The children, then, through imitation or accomodatory activities can answer the question. Here, again, the principle of multiple embodiment is important because the teacher certainly wants the child to initiate such activity in any appropriate situation and may wish to have the children employ imitative behavior in more than one situation, if necessary. The teacher must be sensitive to the type of knowledge the child is acquiring in these imitative activities. The knowledge acquired has a high probability of being physical knowledge for children in the stage of preoperational representation. While this should not alarm the teacher, it would be inappropriate to try to build higher-order concepts on one-to-one correspondence with children at this stage.

Mathematical variability principle. The principle of multiple embodiment is not the only principle the teacher can use in managing play activities of children. It has already been pointed out that while children engage in free play activities, the teacher through artful intervention can change free play activity into a directed activity for the children. If the teacher's suggestion fails to transform the free play activity into a directed activity, she can try other suggestions or a direct demonstration. For example, by placing cowboys and Indians on horses, she can lead the children to engage in imitative activity. Beyond these suggestions, the teacher can employ what is called the mathematical variability principle (Dienes, 1971). In the multiple embodiment principle, the mathematical content is held constant and the materials varied under the constraint of being conducive to construction of the concept by the child. In the mathematical variability principle, the mathematical content is varied. In case of one-to-one correspondence, the teacher can vary the relation being considered to either a new relational cateogry altogether (e.g., length relations; family relations) or vary the relation within the category of matching relations (more than, fewer than, as many as). Each of these variations can be used to create disequilibrium in the child, so that self-regulation is given an opportunity to operate.

Play can vary from free play to directed play. Directed play is a natural extension of free play. The teacher can employ in the context of play the principles of multiple embodiment and mathematical variability in attempting to take a child from physical experience to mathematical experience. But maturation also contributes to the development of certain mathematical concepts; for example, classes, relations, and number. The teacher should not expect dramatic short-term success in teaching those topics to preoperational children. She can expect more success with children in the transitional stages which corresponds here to the second level of mathematical concepts. But again, the short-term success wil undoubtedly be modest. With children in the stage of preoperational representation, it is advocated that the teacher hold the learning phase constant; that is, use the exploratory phase, employ the mathematical variability principle, the multiple embodiment principle, and utilize the free play - directed play distinction. It is felt that it is more humane to utilize a wide variety of mathematical concepts in multiple contexts than to attempt to take the children to the higher two learning instructional phases for particular concepts. This opinion is predicated on the assumption that the preoperational children will be operating at the level of physical experience in most play activities. For the higher two learning instructional phases, the children must be able to acquire mathematical knowledge--which is another way of saying that they must be able to engage in mathematical experience. At the exploratory phase, it is advocated that the mathematical language specific to the mathematical concepts dealt with be developed.

The mathematical variability principle and the multiple embodiment principle both have been explained in terms of the play activities of children. These play activities are viewed as the first learning instructional phase in a cycle of three instructional phases for mathematical concepts. The two others are: a phase of abstraction and representation and a phase of formalization and interpretation.

Abstraction and Representation Phase

The second learning instructional phase identified, that of abstraction and representation, is based in part on the distinction between physical experience and mathematical experience. In the case of physical knowledge, abstraction can and does occur, but it is simple abstraction about properties of objects and generally does not lead to mathematical knowledge. Two examples of such abstractions are hardness and sharpness. But another type of abstraction exists. It is called reflective abstraction which is abstraction from the actions performed on objects or representations of objects. Knowledge gained through reflective abstraction is called mathematical knowledge. An example is the child who counts a string of beads from one end, then from the other and realizes that the number of beads is independent of the order of counting them. The beads are there, but the knowledge gained had to do with the actions of the child. The capability of gaining knowledge came through going back over the actions and realizing their significance or, in other words, reflecting on them. Another example is the child pairing elements from two sets until one is exhausted before the other, and then pairing them again a different way. These actions lead to the realization that it makes no difference how the pairing is done (i.e., one set will always contain more elements than the other). Still another example is the child who compares stick A and stick B, then B and stick C and deduces that A is longer than C based on the two initial comparisons.

The examples are all of situations in which a child may be engaging in play but still engage in reflective abstraction. There is no contradiction here. The child who does engage in reflective abstraction has the potential of going quite beyond play activities in the sense of the exploratory phase. While it is entirely possible to continue in mathematical-like games for such children, such children can engage in much higher level games, insofar as mathematical concepts are concerned. But the teacher need not be restricted to game-like activities in her teaching of these children.

The child can engage in reflective abstraction, but yet, not make a representation of newly gained knowledge. A representation could be a drawing, a diagram, or a collection of symbols. For example, if a child compares a green stick with a red stick and finds the green stick shorter than the red stick, $G < R$ could be used as a representation; $\underset{G}{|} < \underset{R}{\Big|}$ could be used as a representation; or $\underset{G}{|} \quad \underset{R}{\Big|}$ could be used as a representation, among others. If a child is engaged in reflective abstraction and representation, he is definitely operating at a higher level than he was expected to in the exploratory phase.

Formalization and Interpretation Phase

The learning instructional phase of formalization and interpretation completes the cycle of learning mathematical concepts. In order to explicate this phase, the mathematical concept base ten numeration system is selected to illustrate and differentiate the three phases.[2]

We will assume that the child is at the concrete stage of operations in development. Operatory classification and relations are at his disposal. Just because a child is at the stage of concrete operations, he will not necessarily know base ten numeration or even have made representations of his knowledge. What he is able to do with classes, relations, and number has not necessarily been formalized. This knowledge is largely unconscious for the child. However, we will assume the child has completed a learning cycle concerning the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9, can write them, order them, and do simple addition.

Any natural number can be written in expanded notation. For example, 326 in expanded notation is $3 \times 10^2 + 2 \times 10^1 + 6$. The 3, 2, and 6 are called coefficients, 10 is called the base, and the 2 and 1 in 10^2 and 10^1 are called exponents. Consequently, the coefficients, the base, and the exponents can all be allowed to vary in employing the mathematical variability principle. Usually, the base is held constant and only the exponents and coeffieicnts are allowed to vary. This practice is adhered to in all except the very beginnings of instruction in the exploratory phase.

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[2] Subsequent to writing this paper, changes and refinements have been made in the learning and instructional phases for numeration as a result of a teaching experiment.

Imagine that children are given a collection of various assortments of materials, such as geometrical shapes, checkers, or dried beans. The children are allowed to engage in free play with the materials, building whatever they wish -- castles, houses, roads, forts, etcetera. Because these children are at the stage of concrete operations, the teacher can intervene with artful suggestions to direct play activities. The first type of suggestion she could make is to have the children find how many piles with a certain number in each pile they can make. She can employ the mathematical variability principle to vary the number in each pile or the total number of objects in each collection. The total collection should not contain more than, say 40 to 50 objects, or the children will quickly tire of the task. The multiple embodiment principle can also be employed in at least the following ways. The type of objects can be varied, thus setting a new problem each time, or the type of collection formed by the children can be varied. For example, strings of beans with five per string, stacks of blocks with ten per stack, or plates of dried beans with ten per plate can all be used. The essential thing being that a collection of objects can be partitioned into subcollections with the same number in each subcollection and one other subcollection with fewer objects in it than in any other subcollection -- a collection of twenty-six objects can be partitioned into four subcollections with six per subcollection and two more.

In the first few partitions, the children will probably participate at the level of physical experience. One of the first bits of mathematical knowledge the children should acquire is that there is the same number of objects in the total collection before and after partitioning. That is, the child should be able, through his actions, to determine that a pile of objects can always be put back the way it was before the partition, that no objects were added or subtracted, the number of objects before piling is the same as after piling.

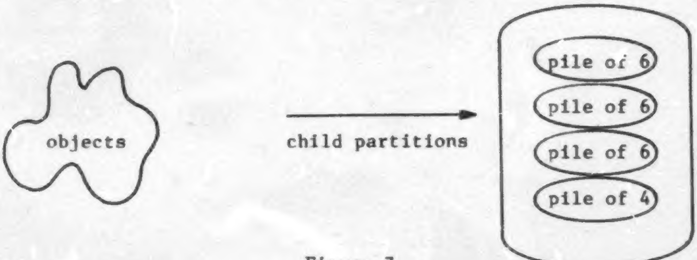

Figure 7

Specifically, if a child makes three piles with six per pile and one pile of four, the child should know that the total number of objects in the original pile is the same as the number of objects in three piles of six and one pile of four, without knowing there are 22 objects. If a child realizes this, he is in transition from the exploratory phase to the abstraction and representation phase. Hopefully, the teacher can use the mathematical variability principle by varying the number of objects in each pile, so that the child will realize that no matter how many are in each pile, the total number in all the piles is the total number of objects.

With this realization, the child is well on his way to constructing the concept of a numeration system. While he has a lot of information yet to acquire, the operational basis for further work has been laid. The basic goal in the second phase is to have the child construct a notational system and construct the place-value concept. The two digit numbers are worked on at different age levels than are the three digit numbers, which in turn are encountered at different age levels than are the four digit and higher digit numbers in order to establish the generalization of place-value.

After the children first enter the second phase, they are able to partition a collection into subcollections and know that the number of objects in the original collection is the same as that in all the subcollections. Capitalizing on this knowledge and the ability of the children to engage in rational counting, place-value concepts may be developed. For example, the children may be given a collection of objects, say 35, and asked to count out a pile of ten and place them in a transparent bag; count out another pile of ten and place them in a transparent bag; and then note there are only five remaining. The children then may fill out the following:________________tens and ______________ones. Various activities such as the above can be done until the children are ready, in the teacher's estimation, to take one further step and write, for example, the symbol "35" to represent 3 tens and 5 ones. Another activity at this second phase which is useful later on in subsequent learning is the tally chart. The tally chart may be thought of as a representation. For two digit numbers, it looks as follows. In the tally chart in Figure 8, the two marks under "tens" mean two sets of ten have been counted, and the mark under "ones" means there is a single element remaining in the total collection. The tally chart is a representation and can be used to represent any number from 0 to 99, inclusive. Use of the tally chart should be coordinated with the development of the numerals. The most singular difficulty children have with the tally chart is that they forget the marks in the tens place have a different meaning than do the marks in the ones place.

tens	ones
//	/

Figure 8

In the activities mentioned, it should be noted that the number names were not mentioned, only the symbols, such as "63" were mentioned, and they were to be interpreted as six tens and three ones. After children can represent any collection, such as a tens and b ones, where a and b are digits, as "ab," they are quite ready to learn the number names for two digit numbers and order the numbers from 0 to 100 The next phase is entered because the knowledge gained to this point is going to by systemized by the children. The basis for the learning has been laid in the counting of piles of ten. However, the main goal of the next learning stage is to systemize the whole numbers from 0 to 100 using the number names. Other learning cycles will be built on this cycle--such as cycles having to do with counting by twos, threes, fours, fives, sixes, etc.; addition, multiplication of two digit numbers and their properties; and subtraction and division.

To initiate the last phase, have the children count out a collection of objects as the basic learning mechanism. When a pile of ten has been found and collected together, then one more ten is counted out. The two tens are symbolized by "20" and the spoken number is given. The decades are developed in this manner and the symbol "<" and the phrase "less than" is introduced or reviewed, as the case may be. The children then should work on ordering the decades arriving eventually at $10 < 20$; $20 < 30$; $30 < 40$; $40 < 50$; $50 < 60$; $60 < 70$; $70 < 80$; $80 < 90$; and any other variation, such as $20 < 70$. The tally chart is useful in developing the above activities. The children should be able to write "$20 < 30$," as well as say "twenty is less than thirty," and be able to complete open sentences such as $\square < 40$ correctly. The mathematical variability principle should be used in all of the activities in the second and third learning phase. The multiple embodiment principle should be used especially in the second phase but to a lesser extent in the third phase. This is natural due to the character of the activities and the level at which they are conducted. In the third phase, however, the multiple embodiment principle is used most in providing interpretations and models of the concept for the child to use.

Formalization is taking place in the sense that a notational system is developed and organized by the child. The organization of the notational system is based on the abstraction and representation accomplished at phase two and on the new element of an order relation. The order relation is an essential part of the third phase for the concept of numeration. Without it, the third phase would have little meaning. The order relation, however, is based on one-to-one correspondence, so that preliminary learning cycles will have to have been completed with regard to one-to-one correspondence and number.

After the decades have been symbolized and ordered, and the child can count by tens, the decades can be completed based on the relation "one more than" just as the decades were ordered on the basis of "one more ten than." Eventually, we want the child to be able to say the number names for all the numbers from 0 to 99 inclusive, be able to write the numerals, order any two of them, have the complete sequence ordered, and be able to count by ones from 1 to 99 inclusive. In the ordering tasks, the child should know, for example, that any number in the "sixties" is greater than any number in the "forties," so that no confusion exists in ordering two numbers such as 47 and 63.

The above learning cycle with regard to numeration will undoubtedly be interrupted by other learning cycles. In fact, it is advocated that two or more learning cycles be operating concurrently so that boredom is decreased.

Summary

For a particular learner, it is assumed that mathematical concepts go through three levels -- essentially no concept, then rudiments of the concept, and then an operational concept. These levels of concepts form the basis for identifying three learning-instructional phases for mathematical concepts -- the exploratory phase, the phase of abstraction and representation, and the phase of formalization and interpretation. These learning-instructional phases interact with the type of experiences and the cognitive stage of the child. Mathematical concepts which have been shown to be developmental in nature (number, relation, and classes) need to be considered different than concepts which have not been shown to be developmental in nature. For the case of the latter category of concepts (numeration, addition, subtraction, multiplication), if a child is preoperational, then the teacher should not force the child to go beyond the exploratory learning-instructional phase. The only type of experience such a child is capable of is physical experience. It must be stated explicitly that the type of experience of a child is not under the control of the teacher. She can give children the opportunity to engage in mathematical experiences, but there is no way she can force the child to engage in mathematical experiences. Moreover, the child has little or no conscious control over which type of experience he engages in. It is something which just happens and, to a large extent, depends on the cognitive stage which the child is in. So, for preoperational children, the teacher should not expect the children to go beyond physical experience in the exploratory phase. She can aid the child (or give the child the opportunity) to engage in mathematical experience through employment of the multiple embodiment principle and the mathematical variability principle, but she cannot make the reflective abstraction for the child. Through the process of self-regulation, the child will eventually realize the significance of his actions and thus enter the next learning-instructional phase with regard to particular concepts. Through maturational processes and experience, preoperational children will move to the concrete operational stage and thus become much more likely to engage in reflective abstraction.

Concrete operational children have much more mathematical learning potential than do preoperational children. These children will begin, with regard to particular concepts, with play activities just as do the children in the two other phases. While they may begin with physical experience, they can go quite beyond physical to mathematical experience, and hence to the higher two learning-instructional phases. If a child is operating at the phase of abstraction and representation, with regard to a particular concept, there is no way that he will engage only in physical experience because of the very definition of that phase. The same can be said for the next higher phase.

All children, then, whether they are concrete operational, preoperational, or transitional, should be given the opportunity to learn the same mathematical concepts. The depth of concept learning will depend to a large extent on his stage of development -- so that a teacher should expect concrete operational children to complete the learning cycles. But children in lower stages of cognitive development should not be expected to complete the learning cycles; however, they should be given the opportunity to do so.

One further comment is in order concerning physical experience and mathematical experience. One should not equate physical experience with physical objects, or physical actions on such objects, and mathemtical experience with abstract thought. A child may engage in mathematical experience through manipulation of physical objects, physically or mentally. The presence or absence of the objects just does not determine whether a child engages in mathematical experience. In fact, presence of objects may facilitate a mathematical experience -- but the presence of objects is not logically necessary for a mathematical experience. Physical actions are not necessary for a child to engage in physical experience -- but some physical context is. So, whether a child engages in physical experience or mathematical experience depends ultimately on the child.

Tabular Representation of a Learning-Instructional Model

Types of Experience

L-I Phase / Cognitive Stage	Exploratory	Abstraction and Representation	Formalization and Interpretation
Concrete Operational	Physical and Mathematical	Physical and Mathematical	Physical and Mathematical
Transitional	Physical and Mathematical	Physical and Mathematical	
Preoperational	Physical		

Figure 9

In the tabular representation of the learning-instructional model (see Figure 9), concrete operational children are capable, at each learning-instructional phase, of engaging in physical or mathematical experience. It must be emphasized that a child's being in the stage of concrete operations does not guarantee he will be able to complete a learning-instructional cycle with each concept presented. However, it would be unlikely that a child who did complete a learning-instructional cycle would be a preoperational or transitional stage child, especially a preoperational child. In fact, it is not assumed that a child in the preoperational stage is capable of making a reflective abstraction. Children in the transitional stage should be expected to be capable of making abstractions and representations, but not necessarily be capable of organizing their knowledge in the sense of the formalization and interpretation phase. Preoperational children should be expected to be capable of engaging in mathematical experiences. It should be emphasized, however, that there may be exceptions to the patterns outlined in the tabular representation.

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Participants

1. Joe Austin
2. Harry Beilin
3. Mary Ann Byrne
4. Thomas Cooney
5. Suzanne Damarin
6. Jon Englehardt
7. William Geeslin
8. E. Glenadine Gibb
9. Herbert Ginsburg
10. Lars Jansson
11. David Johnson
12. J. Larry Martin
13. Harold Mick
14. Doyal Nelson
15. Carol Novillis
16. Thomas O'Brien
17. Thomas Post
18. Robert Reys
19. Donald Saari
20. John Schleup
21. Richard Shumway
22. Charles Smock
23. Leslie Steffe
24. H. Laverne Thomas
25. A. Edward Uprichard